Available online at www.sciencedirect.com

SCIENCE DIRECT

J. Differential Equations 196 (2004) 289–300

Journal of Differential Equations

http://www.elsevier.com/locate/jde

Hyperbolicity, transversality and analytic first integrals

Jacky Cresson

Equipe de Mathématiques de Besançon CNRS-UMR 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon, Cedex, France

Received October 25, 2000; revised August 11, 2003

Abstract

Let A be a (normally) hyperbolic compact invariant manifold of an analytic diffeomorphism f of an analytic manifold M. We assume that the stable and unstable manifold of A intersect transversally (in an admissible way), the dynamics on A is ergodic and the modulus of the eigenvalues associated to the stable and unstable manifold, respectively, satisfy a non-resonance condition. In the case where A is a point or a torus, we prove that the discrete dynamical system associated to f does not admit an analytic first integral. The proof is based on a triviality lemma, which is of combinatorial nature, and a geometrical lemma. The same techniques, allow us to prove analytic non-integrability of Hamiltonian systems having Arnold diffusion. In particular, using results of Xia, we prove analytic non-integrability of the elliptic restricted three-body problem, as well as the planar three-body problem.

Keywords: Normally hyperbolic manifold; Analytic first integral; Partially hyperbolic tori; Three-body problem

1. Introduction

The aim of this paper is to discuss the following conjecture [2]

Conjecture 1.1. *Let f be an analytic diffeomorphism of an analytic manifold M, and A be a compact hyperbolic invariant set for f. We assume that*

(i) *the stable and unstable manifold of A intersect transversally,*
(ii) *f is ergodic on A,*
(iii) *the eigenvalues of f associated to the stable (resp. unstable) manifold satisfy a non-resonance condition.*

E-mail address: cresson@math.univ-fcomte.fr.

doi:10.1016/j.jde.2003.10.002

Then the discrete dynamical system associated to f does not admit an analytic first integral, except constant.

In this paper, we prove this conjecture for a point or a torus.

The proof is based on two results. The first one, called *triviality lemma*, states that under assumption (ii) and (iii), an analytic function, which is zero on a generic orbit of the stable (or unstable) manifold, is identically zero. Then, conditions (ii) and (iii) are sufficient conditions under which the stable and unstable manifold of a point, or a torus, are a key-set for analytic functions.

The second result, called *geometrical lemma*, states that a C^1 function, constant on the stable and unstable manifold has a differential which is zero at all point of transverse intersection.

The same techniques allow us to prove that a Hamiltonian system H possessing a partially hyperbolic torus satisfying assumption (i)–(iii) does not admit an analytic first integral independent of H. This result implies, via Xia study of Arnold diffusion in the three-body problem, non-existence of analytic first integrals for the elliptic restricted three-body problem, as well as the planar three-body problem, extending a well-known result of Poincaré.

2. Hyperbolic fixed point

2.1. On a theorem of Moser

Let f be an analytic diffeomorphism of $\mathbb{R}^n$. We say that f possesses a *transverse hyperbolic homoclinic structure* if f admits an invariant hyperbolic fixed point p, whose stable and unstable manifolds, denoted $W^-(p)$ and $W^+(p)$, intersect transversally.

In 1973, Moser [5] proves, for $n = 2$, the following theorem.

Theorem 2.1. *Let f be an analytic diffeomorphism of* $\mathbb{R}^2$, *possessing a transverse hyperbolic homoclinic structure, then the dynamical system associated to f does not admit an analytic first integral.*

His proof is based on the Birkhoff–Smale theorem. Precisely, he uses the existence of a hyperbolic invariant set in the neighbourhood of the homoclinic orbit, on which the dynamics is complicated. In particular, there exists a dense orbit. This set is then a key-set for analytic functions.

The generalization of this result in higher dimension is difficult (see [3]) if one wants to follow Moser's scheme of proof. This is due in particular, to the fact that key's sets of analytic functions with several variables are complicated to characterize.

2.2. Main result

Let f be a diffeomorphism of $\mathbb{R}^n$, possessing a transverse hyperbolic homoclinic structure.

We call *local analytic first integral* for f, a C^1 first integral, such that its restriction to an open neighbourhood $\mathcal{U}$ of $W^-(p) \cup W^+(p)$ is analytic.

Remark 2.1. This definition has been suggested by R. Roussarie in order to cover some problems concerning first integrals of polynomial vector fields.

We say that the local analytic first integral is C^ω-trivial, if its restriction to $\mathcal{U}$ is constant.

Theorem 2.2. *Let f be an analytic diffeomorphism of $\mathbb{R}^n$ such that p is a hyperbolic fixed point for f. We assume that*

(i) *$W^-(p)$ and $W^+(p)$ intersect transversally in an admissible homoclinic point h,*
(ii) *the eigenvalues of $Df(p)$ associated to $W^-(p)$ (resp. $W^+(p)$), denoted λ_i^-, $i = 1, \dots, n^-$ and $\lambda_i^+, i = 1, \dots, n^+$, respectively, satisfy the following non-resonance condition:*

$$|(\lambda^\sigma)^\nu| \neq 1 \tag{1}$$

for $\sigma = \pm$, *where* $\nu \in \mathbb{Z}^{n^\sigma} \setminus \{0\}, \nu = (\nu_1, \dots, \nu_{n^\sigma}), \lambda^\sigma = (\lambda_1^\sigma, \dots, \lambda_{n^\sigma}^\sigma)$, $(\lambda^\sigma)^\nu = (\lambda_1^\sigma)^{\nu_1} \dots (\lambda_{n^\sigma}^\sigma)^{\nu_{n^\sigma}}$.

Then, the dynamical system defined by f does not possess an analytic first integral which is not C^ω-trivial.

The notion of *admissible* homoclinic point will be precised during the proof of the theorem (see Definition 2.1).

For diffeomorphisms of $\mathbb{R}^2$, the non-resonance condition is empty, as well as the condition on the homoclinic point to be admissible. Then, if we look for an analytic first integral defined on the whole space, the theorem implies that it is trivial. As a consequence, the theorem of Moser is a corollary of our result.

2.3. Proof of Theorem 2.2

2.3.1. Preliminary

The proof of Theorem 2.2 is based on two key results. The first one is of *combinatorial* nature, and is related to the dynamics on the stable or unstable manifold (the triviality lemma). The second one, if of *geometrical* nature, and is related to the transverse structure in each iterates of the homoclinic point.

Let $x_0 \in \mathbb{R}^n$, we denote $\gamma(x_0)$ the orbit of x_0 under f.

Lemma 2.1 (Triviality lemma). *Let f be an analytic diffeomorphism of $\mathbb{R}^n$ satisfying assumption* (ii) *of Theorem* 2.2. *Let A be an analytic function on $W^-(p)$ (resp. $W^+(p)$) such that $A(x) = 0$ for all $x \in \gamma(x^-)$ (resp. $x \in \gamma(x^+)$), where x^- (resp. x^+) is an admissible point of $W^-(p)$ (resp. $W^+(p)$), then $A = 0$ on $W^-(p)$ (resp. $W^+(p)$).*

The proof is given in the next section.

Lemma 2.2 (Geometrical lemma). *Let f be an analytic diffeomorphism of $\mathbb{R}^n$ satisfying the assumptions of Theorem* 2.2. Let A be a function of class C^1 *which is constant on $W^-(p) \cup W^+(p)$, then $DA(x) = 0$ for all $x \in \gamma(h)$.*

The proof is given in appendix.

2.3.2. Proof

Let P be an analytic first integral for f. The idea is to prove by induction, the cancellation of the successive derivatives of P, denoted $DP^i(x)$, for all $x \in \gamma(h)$ where h is an admissible point. As P is analytic on $\mathcal{U}$ and $W^-(p) \cup W^+(p) \subset \mathcal{U}$ which is a connected set, we deduce that $P = \text{const}$ on $\mathcal{U}$.

The induction is based on the following property.

(h_n) We have $DP^i(x) = 0$ for all $x \in \gamma(h)$, and $1 \leqslant i \leqslant n$.

This property is satisfied for $n = 1$. Indeed, we have $P(x) = \text{const}$ on $W^-(p) \cup W^+(p)$ by definition. The geometrical lemma implies $DP(x) = 0$ for all $x \in \gamma(h)$.

We now prove that (h_n) implies (h_{n+1}): By (h_n), we have $DP^n(x) = 0$ for all $x \in \gamma(h)$. By the triviality lemma, we deduce that $DP^n|_{W^-(p)} = 0$ and $DP^n\ |_{W^-(p)} = 0$. Then, by the geometrical lemma, we obtain $DP^{n+1}(x) = 0$ for all $x \in \gamma(h)$.

By induction, we then have $DP^i(x) = 0$ for all $x \in \gamma(h)$ and $i \geqslant 1$, which concludes the proof of the theorem. $\square$

2.4. Proof of the triviality lemma

2.4.1. Reduction to a linear diffeomorphism

Let $f^-(x)$ be the restriction of f to $W^-(p)$. The linear map $Df^-(p)$ admits eigenvalues λ_i^-, $i = 1, \dots, n^-$, such that $0 < |\ \mathrm{Re}(\lambda_i^-)\ | < 1$ for $i = 1, \dots, n^-$. Moreover, by the non-resonance condition (ii) of Theorem 2.2, we have the Poincaré theorem [1, p. 186], for analytic linearization of f^-. There exists an analytic coordinates system $y = z(x)$, defined on an open neighbourhood U of p in $W^-(p)$, such that $z \circ f^- \circ z^{-1} = f_{\text{lin}}^-$, where $f_{\text{lin}}^-(x) = Df^-(p).x$. We denote by x^- the image of h in this coordinates system.

As $A((f^-)^k(h)) = \text{const}$ by assumption, we have $A \circ z^{-1} \circ (f_{\text{lin}})^k \circ z(x^-) = \text{const}$. We denote $\tilde{A} = P \circ z^{-1}$ and $y^- = z(x^-)$, then

$$\tilde{A}((f_{\text{lin}}^-)^k(y^-)) = \text{const}.$$

The function $\tilde{A}$ is still analytic on U. If $\tilde{A} \equiv 0$ on U, then $A \equiv 0$ on $z^{-1}(U)$. As $z^{-1}(U)$ an open neighbourhood of $W^-(p)$, and $W^-(p)$ is a connected set, we have $A \equiv 0$ on $W^-(p)$.

We can always find an open set $V \subset U$, containing p, such that $\tilde{A}(x) = \sum_\nu a_\nu x^\nu$, for all $x \in V$.

Moreover, we can find an holomorphic coordinates system which diagonalizes f^-_{lin}. We denote by $F^-_{\mathrm{lin}} : \mathbb{C}^{n^-} \to \mathbb{C}^{n^-}$ the linear mapping defined by $F^-_{\mathrm{lin}}(x) = \Lambda^-.x$, where Λ^- is a diagonal matrix, constituted of the eigenvalues λ^-_i, $i = 1, \dots, n^-$. We denote by h^- the image of h in this coordinates system. A similar reasoning for $W^+(p)$ produces a point h^+.

Definition 2.1. A point $h \in W^-(T)$ (resp. $W^+(T)$) is called admissible if h^- (resp. h^+) belongs to $(\mathbb{C}^*)n^-$ (resp. $(\mathbb{C}^*)^{n^+}$). A homoclinic point h is called admissible if h^+ and h^- belongs to $(\mathbb{C}^*)^{n^-}$ and $(\mathbb{C}^*)^{n^-}$, respectively.

Remark 2.2. We do not know if admissible points are generic in the analytic category.

By the previous remarks, the triviality lemma follows from the following lemma.

Lemma 2.3. *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be a linear mapping defined by $F(x) = \Lambda.x$, where Λ is a diagonal matrix whose eigenvalues $\lambda_i, i = 1, \dots, n$, satisfy the non-resonance condition and are such that $|\lambda_i| < 1$ for $i = 1, \dots, n$ (resp. $|\lambda_i| > 1$ for $i = 1, \dots, n$). Let A be an holomorphic function and h a point in $(\mathbb{C}^*)^n$. If $A(F^k(h)) = 0$ for all $k \in \mathbb{N}$ then $A \equiv 0$.*

The proof is detailed in the next paragraph.

2.4.2. Proof of Lemma 2.3

We give the proof for a contracting mapping. The case of an expansive mapping is similar.

Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be of the form

$$F(x) = (\lambda_1 x_1, \dots, \lambda_n x_n),$$

where $0 < |\lambda_i| < 1$ for $i = 1, \dots, n$ and satisfy the non-resonance condition. Let $h = (h_1, \dots, h_n) \in (\mathbb{C}^*)^n$, and

$$A(x) = \sum_{v \in \mathbb{N}^n} a_v x^v,$$

$v = (v_1, \dots, v_n)$, $x^v = x_1^{v_1} \dots x_n^{v_n}$, an holomorphic function. We denote $|v| = v_1 + \dots + v_n$.

We have $A(F^k(h)) = 0$ for all $k \in \mathbb{N}$ by assumption, then

$$a_0 + \sum_v a_v \lambda^{kv} h^v = 0, \quad \forall k \in \mathbb{N}. \tag{2}$$

As the eigenvalues $(\lambda_1, \dots, \lambda_n)$ satisfy a non-resonance condition, the quantities $|\lambda^v|$, $v \in \mathbb{N}^n$ can be *totally ordered*, i.e.

$$|\lambda^{v_0}| > |\lambda^{v_1}| > \dots > |\lambda^{v_k}| > \dots .$$

As a consequence, Eq. (2) is equivalent to

$$\sum_{i \geqslant 0} a_{v_i} \lambda^{k v_i} h^{v_i} = 0, \quad \forall k \in \mathbb{N}. \tag{3}$$

The cancellation of A is done by induction. We first factorize Eq. (3) by λ^{v_0}. We obtain

$$a_{v_0} h^{v_0} + \sum_{i \geqslant 1} a_{v_i} \left(\frac{\lambda^{v_i}}{\lambda^{v_0}} \right)^k h^{v_i} = 0, \quad \forall k \in \mathbb{N}. \tag{4}$$

As $|\lambda^{v_i}/\lambda^{v_0}| < 1$ for all $i \geqslant 1$, taking the limit of (4) when $k \to \infty$, we obtain

$$a_{v_0} h^{v_0} = 0. \tag{5}$$

As h is an admissible point, we have $h^{v_0} \neq 0$ and $a_{v_0} = 0$.

A simple induction on i allows us to prove that $A \equiv 0$. This concludes the proof of the lemma. □

3. Normally hyperbolic tori

Let f be an analytic diffeomorphism of an analytic manifold M, and T an invariant, n-dimensional normally hyperbolic torus for f. Following [7, p. 322], there exists an analytic coordinates system, defined in a neighbourhood U of T, such that f takes the form

$$f(\theta, s, u) = (\theta + 2\pi\omega, \Lambda^+(\theta)s, \Lambda^-(\theta)u) + r(\theta, s, u), \tag{6}$$

where $(\theta, s, u) \in \mathbb{T}^n \times \mathbb{R}^{l^-} \times \mathbb{R}^{l^+}$, r is of order 2 in s and u, and $r(\theta, 0, u) = 0$, $r(\theta, s, 0) = 0$.

In this coordinates system, the invariant torus T is given by

$$T = \{(\theta, s, u) \in \mathbb{T}^n \times \mathbb{R}^{l^-} \times \mathbb{R}^{l^+} | s = u = 0\}$$

and its stable and unstable manifolds are given by

$$W^-(T) = \{(\theta, s, u) \in \mathbb{T}^n \times \mathbb{R}^{l^-} \times \mathbb{R}^{l^+} | u = 0\},$$

$$W^+(T) = \{(\theta, s, u) \in \mathbb{T}^n \times \mathbb{R}^{l^-} \times \mathbb{R}^{l^+} | s = 0\}.$$

respectively.

The torus T is said to be *reducible* if the matrices $\Lambda^\sigma(\theta)$, $\sigma = \pm$, are *independent* of θ.

Definition 3.1. An homoclinic point h to T is called admissible if for some iterates, we have $f^{n^-}(h) = (\theta^-, s^-, 0) \in W^-(T)$ and $f^{n^+}(h) = (\theta^+, 0, u^+)$ with $s^- \in (\mathbb{R}^*)^{l^-}$ and $u^+ \in (\mathbb{R}^*)^{l^+}$.

We have the following theorem.

Theorem 3.1. *Let T be a reducible invariant normally hyperbolic torus of an analytic diffeomorphism f. We assume that*

(i) *the stable and unstable manifold intersect transversally in an admissible homoclinic point h,*
(ii) *the dynamics on the torus is minimal,*
(iii) *the eigenvalues associated to the stable and unstable manifold, denoted by λ_i^σ, $i = 1, \dots, l^\sigma$ satisfy the non-resonance condition*

$$(|\lambda_i^\sigma|)^\nu \neq 1,$$

for all $\nu \in \mathbb{Z}^{l\sigma} \backslash \{0\}$, $\sigma = \pm$.

Then, the discrete dynamical system defined by f does not admit an analytic first integral.

The scheme of proof is similar to that of Theorem 2.2. The geometrical lemma can be applied. We only need to prove the following version of the triviality lemma.

Lemma 3.1 (Triviality lemma for normally hyperbolic tori). *Let T be a reducible normally hyperbolic torus. Let h^+ (resp. h^-) be an admissible point of $W^+(T)$ (resp. $W^-(T)$), and A an analytic function, which vanishes on the orbit $\gamma(h^+)$ (resp. $\gamma(h^-)$) of h^+ (resp. h^-). If the modulus of the eigenvalues of Λ^+ (resp. Λ^-) satisfy the non-resonance condition and the flow on T is minimal, then $A \equiv 0$ on $W^+(T)$ (resp. $W^-(T)$).*

Proof. We detail the proof for $W^-(T)$. The proof is similar for $W^+(T)$.

Let A be an analytic function on $W^-(T)$. In a sufficiently small neighbourhood V of T, A takes the form

$$A(\theta, s) = \sum_{k \in \mathbb{N}^{l^-}} a_k(\theta) s^k, \tag{7}$$

where $a_k(\theta)$ is a 2π periodic function of θ.

Let $h^- = (\theta^-, s^-) \in U \cap W^-(T)$ such that $s^- \neq 0$. The restriction of f to $W^-(T)$ is defined in $U \cap W^-(T)$ by

$$f^-(\theta, s) = (\theta + \omega, \Lambda^- s). \tag{8}$$

By assumption, we have $A((f^-)^m(h^-)) = 0$ for all $m \in \mathbb{N}$, hence

$$\sum_{k \in \mathbb{N}^{l^-}} a_k(\theta^- + m\omega)(\lambda^-)^{mk}(s^-)^k = 0, \quad m \in \mathbb{N}. \tag{9}$$

As the eigenvalues λ^- satisfy a non-resonance condition, the quantities $(\lambda^-)^k$, $k \in \mathbb{N}^{l^-}$, are totally ordered, i.e.

$$(\lambda^-)^{k_0} > (\lambda^-)^{k_1} > \cdots > (\lambda^-)^{k_i} > \cdots. \tag{10}$$

As a consequence, Eq. (9) is equivalent to

$$\sum_{i \geqslant 0} a_{k_i}(\theta^- + m\omega)(\lambda^-)^{mk_i}(s_-)^{k_i} = 0, \quad \forall m \in \mathbb{N}. \tag{11}$$

The cancellation of A is done by induction on i. We factorize $(\lambda^-)^{mk_0}$ in Eq. (11). We obtain

$$a_{k_0}(\theta^- + m\omega)(s^-)^{k_0} + \sum_{i \geqslant 1} a_{k_i}(\theta^- + m\omega)\left(\frac{(\lambda^-)^{k_i}}{(\lambda^-)^{k_0}}\right)^m (s^-)^{k_i} = 0, \quad \forall m \in \mathbb{N}. \tag{12}$$

As $|(\lambda^-)^{k_i}/(\lambda^-)^{k_0}| < 1$ for all $i \geqslant 1$, and $s^- \in (\mathbb{C}^*)^{l^-}$, we have, taking the limit of (12) when $m \to \infty$,

$$\lim_{m \to \infty} a_{k_0}(\theta^- + m\omega) = 0. \tag{13}$$

As ω is non-resonant, we deduce, by a density argument, that $a_{k_0}(\theta) = 0$ for all $\theta \in \mathbb{T}^n$.

A simple induction on i concludes the proof. $\square$

4. Partially hyperbolic tori

Let M be an analytic symplectic manifold of dimension $2n + 2l$, and H an analytic Hamiltonian system defined on M. We call *partially hyperbolic torus*, an invariant n-dimensional torus, for which there exists a neighbourhood such that the Hamiltonian takes the form

$$H(\phi, I, s, y) = \tilde{\omega}.I + \tfrac{1}{2} I.\Gamma f + x.\Pi y + g(\phi, I, x, y),$$

where $(\phi, I, x, y) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l$, with the usual scalar product, Γ and Π two symmetricals matrices, and g is of order 3 in (I, x, y).

We assume that $\tilde{\omega}$ satisfies a diophantine condition

$$|\tilde{\omega}.k| \geqslant \frac{\gamma}{|k|^{\tau}} \tag{14}$$

for all $k \in \mathbb{Z}^n \backslash \{0\}$, $\gamma > 0$ and $\tau > 1$.

The partially hyperbolic torus T is defined by

$$T = \{(\phi, I, x, y) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l | I = x = y = 0\}.$$

Its stable manifold (resp. unstable manifold), denoted by $W^-(T)$ (resp. $W^+(T)$), is defined by

$$W^-(T) = \{(\phi, I, x, y) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l | I = y = 0\},$$

$$(\text{resp.} W^+(T) = \{(\phi, I, x, y) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l | I = x = 0\}).$$

As $\tilde{\omega}$ is non-resonant, there exists a $2n$-dimensional Poincaré section S, defined in a neighbourhood of T, such that the first return map takes the form

$$f(\theta, \rho, s, u) = (\theta + 2\pi\omega + v\rho, \rho, \Lambda s, \Lambda^{-1}u) + r(\theta, \rho, s, u), \tag{15}$$

where $(\theta, \rho, s, u) \in \mathbb{T}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}$, r is of order 2 in ρ, s and u, ω is non-resonant, and Λ is a diagonal matrix with real eigenvalues λ_i, $i = 1, \dots, l$.

Remark 4.1. Invariant partially hyperbolic tori of near integrable Hamiltonian systems, obtained by bifurcation of resonant tori of integrable Hamiltonian systems along simple resonance ($l = 1$), possess a first return map of form (15). For $l > 1$, this form is valid only under particular conditions of reductibility of the flow on the torus [6].

Theorem 4.1. *Let T be an invariant l-partially hyperbolic torus of an analytic Hamiltonian system $H, l \geqslant 1$. Let $\mathscr{H}$ be the energy level containing T. We assume that*

(i) *the stable an unstable manifold of T intersect transversally in $\mathscr{H}$,*
(ii) *the eigenvalues λ_i, $i = 1, \dots, l$, satisfy the non-resonance condition*

$$\lambda^{\nu} \neq 1, \tag{16}$$

where $\nu \in \mathbb{Z}^l \backslash \{0\}$, $\nu = (\nu_1, \dots, \nu_l)$, $\lambda^{\nu} = \lambda_1^{\nu_1} \dots \lambda_l^{\nu_l}$.

Then, the Hamiltonian system does not admit an analytic first integral independent of H.

The proof is similar to the normally hyperbolic case. We are then reduce to prove the triviality lemma for partially hyperbolic tori.

Lemma 4.1 (Triviality lemma for partially hyperbolic tori). *Let T be a partially hyperbolic torus. Let h^+ (resp. h^-) be an admissible point of $W^+(T)$ (resp. $W^-(T)$), and A an analytic function, which vanishes on the orbit $\gamma(h^+)$ (resp. $\gamma(h^-)$) of h^+ (resp. h^-). If the modulus of the eigenvalues of Λ satisfy the non-resonance condition and the flow on T is minimal, then $A \equiv 0$ on $W^-(T)$ (resp. $W^+(T)$).*

Proof. We prove the lemma for the stable manifold, the case of the unstable manifold being similar. The stable manifold is defined by $W^-(T) = \{(\theta, I, s, u) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^l | I = u = 0\}$. An analytic function on $W^-(T)$ is then of the form $A(\theta, s) = \sum_{k \in \mathbb{N}^n} a_k(\theta) s^k$. As the dynamics on $W^-(T)$ is of the form $f^-(\theta, s) = (\theta + 2\pi\omega, \Lambda s)$, we must solve an equation similar to (9). □

5. The three-body problem and Arnold diffusion

The elliptic restricted three-body problem is the study of the behaviour of a particle A, of mass zero, in Newtonian interaction with two points J and S, of mass $\mu \in]0, 1]$ and $1 - \mu$, respectively, such that the vector SJ describes an ellipse, with eccentricity e and focus at the centre of mass.

The Hamiltonian of this system is given by

$$H_{e,\mu}(t, q, p) = \frac{||p||^2}{2} - \left(\frac{\mu}{\delta(t, q, e)} + \frac{1 - \mu}{\sigma(t, q, e)} \right), \tag{17}$$

where $(t, q, p) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$, $|| \cdot ||$ is the Euclidean norm, $\delta(t, q, e) = ||q - J_t||$, $\sigma(t, q, e) = ||q - S_t||$ with $S_t = ((1 - \mu) r \cos u, (1 - \mu) r \sin u)$, $J_t = (-\mu r \cos u, \mu r \sin u)$, $r = \frac{1 - e^2}{1 + e \cos u}$ and $u = e \sin u + \frac{t}{\sqrt{1 - e^2}}$.

In [8], Xia proves, in his study of Arnold diffusion in the three-body problem, the following theorem.

Theorem 5.1. *For $0 < e \ll 1$ and $0 < \mu \ll e$, there exists invariant 1-partially hyperbolic tori for $H_{e,\mu}$. Let T be such a torus. We denote by $\mathscr{H}_{e,\mu}$ the energy level containing T. The stable and unstable manifold of T intersect transversally in $\mathscr{H}_{e,\mu}$.*

The first return map defined in a neighbourhood of Xia tori is of form (15). Moreover, the dynamics on each of these tori is minimal. As they are 1-hyperbolic, the non-resonance condition, as well as the genericity of the homoclinic point are always satisfied. Then, Theorem 4.1 applies, and we have:

Theorem 5.2. *The elliptic restricted three-body problem does not admit an analytic first integral independent of $H_{e,\mu}$, for $0 < e \ll 1$ and $0 < \mu \ll e$.*

This theorem is announced by Xia in [8] without proof. This result extend to the planar three-body problem using Xia work [9].

6. Conclusion

The proof of Theorem 3.1 extends immediately to the following case with minor modifications.

Theorem 6.1. *Let A be a normally hyperbolic manifold of an analytic diffeomorphism f defined on an analytic manifold M. We assume that there exists an analytic coordinates systems $(x,s,u)\in A\times\mathbb{R}^l\times\mathbb{R}^n$, defined on an open neighbourhood of A such that f is of the form*

$$f(x,s,u) = (g(x), \Lambda^- s, \Lambda^+ u) + r(x,s,u), \tag{18}$$

where Λ^σ, $\sigma = \pm$ are diagonal matrices, r is of order 2 in s and u, and $g: A \to A$ is a diffeomorphism.

We assume that

(i) *the stable and unstable manifold of A intersect transversally in M. There exists an admissible homoclinic point h in $W^-(A)\cap W^+(A)$,*
(ii) *g is ergodic on A,*
(iii) *the eigenvalues λ^σ of Λ^σ, $\sigma=\pm$ satisfy a non-resonance condition.*

Then, f does not admit an analytic first integral except constant.

We follow the proof of Theorem 3.1. The analogue of Eq. (11) is

$$\sum_{i\geqslant 0} a_{k_i}(g^m(x^-))(\lambda^-)^{k_i}(s_-)^{k_i} = 0, \quad \forall m\in\mathbb{N}, \tag{19}$$

where $(x^-, s^-, 0)$ are the coordinates of some iterates of the homoclinic point h. We deduce

$$\lim_{m\to\infty} a_{k_0}(g^m(x^-)) = 0. \tag{20}$$

As g is ergodic, a density argument implies $a_{k_0}(x) = 0$ for all $x\in A$. A simple induction on i concludes the proof.

Theorem 6.1 is then a first step toward the conjecture.

However, in order to cover a more general situation, we must deal with *non-reducible* normally hyperbolic manifolds, i.e. the normal form (18) is replaced by

$$f(x,s,u) = (g(x), \Lambda^-(x)s, \Lambda^+(x)u) + r(x,s,u). \tag{21}$$

In this case, the analogue of Eq. (11) is very complicated. Even in the (non-generic) case of diagonal matrices $\Lambda^\sigma(x)$, $\sigma=\pm$, we must use *Oseledec multiplicativ ergodic theorem* (see [4, p. 665]) in order to conclude.

Appendix. The geometrical lemma

The intersection between $W^-(p)$ and $W^+(p)$ is transversal in a space of dimension n, with $W^-(p) = n^-$ and $\dim W^+(p) = n^+$, $n = n^- + n^+$. The geometrical lemma is a consequence of the following lemma.

Lemma A.1 (General geometrical lemma). *Let M be an n-dimensional manifold of class C^k, $k \geqslant 1$, and V^- (resp. V^+) a n^--submanifold (resp. n^+ -submanifold) of class C^k, such that V^- and V^+ intersect transversally in M. Let P a function on M of class C^1, constant on $V^- \cup V^+$, then $DP(x) = 0$ for $x \in V^+ \cap V^-$.*

Proof. As V^- and V^+ intersect transversally, the tangent bundle in $x \in V^+ \cap V^-$ is $T_x M = T_x V^- + T_x V^+$. As P is constant on V^- (resp. V^+), $DP(x)$ is zero on $T_x V^-$ (resp. $T_x V^+$), then identically zero. □

References

[1] V.I. Arnold, Chapitres supplémentaires de la théorie des équations différentielles ordinaires, Ed. Mir-Librairie du globe, 1996.
[2] J. Cresson, Hyperbolicité et non-intégrabilité analytique II. Tores normalement et partieliement hyperboliques, C. R. Acad. Sci. Paris, Serie I 333 (2001) 229–232.
[3] S. Dovbysh, Transversal intersection of separatrices and branching of solutions as obstructions to the existence of an analytic integral in many-dimensional systems I, Basic result, sepraratrices of hyperbolic points, Collectanea Math. 50 (2) (1999) 119–197.
[4] A. Katok, B. Hassenblatt, Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, Cambridge, 1995.
[5] J. Moser, Stable and random motions in dynamical systems (with a special emphasis in celestial mechanics), in: Ann. Math. Stud., Vol. 77, Princeton University Press, Princeton, NJ, 1973.
[6] D. Treshchev, The mechanism of destruction of resonance tori for Hamiltonian systems, Math. USSR Sbornik 68 (1991) 181–203.
[7] S. Wiggins, Global Bifurcation and Chaos, Springer, Berlin, 1988.
[8] J. Xia, Arnold diffusion in the elliptic restricted three-body problem, J. Dynamics Differential Equations 5 (2) (1993).
[9] J. Xia, Arnold diffusion and oscillatory solutions in the planar three-body problem, J. Differential Equations 110 (1994) 289–321.