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Scale relativity theory for one-dimensional non-differentiable manifolds

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Abstract

We discuss a rigorous foundation of the pure scale relativity theory for a one-dimensional space variable. We define several notions as "representation" of a continuous function, scale law and minimal resolution. We define precisely the meaning of a scale reference system and space reference system for non-differentiable one-dimensional manifolds. © 2002 Elsevier Science Ltd. All rights reserved.

1. Introduction

The aim of this paper is to discuss a rigorous foundation of the *scale relativity theory* developed by Nottale [14,15].

Nottale's fundamental idea is to give up the hypothesis of differentiability of the space-time continuum. Previous works in this direction have been done by Ord [1] [17]. As a consequence, one must consider continuous but non-differentiable objects, i.e., a *non-differentiable manifold.* [2] This leads naturally to the notion of *fractal* functions and manifolds.

In order to justify Nottale's framework, we must develop an *analysis* on non-differentiable manifolds. This analysis does not exist already. However, Nottale's approach to fractal space-time can be used to begin such a work.

In the following, we define a scale and space reference system for graphs of non-differentiable real-valued functions. The difficulty in defining an intrinsic coordinates system on the graph Γ of a non-differentiable function, f, comes from the fact that one cannot define classical curvilinear coordinates. Indeed, a consequence of Lebesgue's theorem is that the length of every part of Γ is infinite. One can overcome this difficulty using representation theory, i.e., associating to f a one-parameter family of differentiable functions $F(t;\epsilon)$, $\epsilon > 0$, such that $F(t;\epsilon) \to f(t)$ when $\epsilon \to 0$. For all $\epsilon > 0$, we then define curvilinear coordinates. The study of Γ is then reduced to the study of the family $F(t;\epsilon)$ when ϵ varies. This leads to several new concepts like scale laws and minimal resolution.

Using all these notions, we can justify a part of Nottale's work on scale relativity.

A general programme to study non-differentiable manifolds is discussed in [6] which leads to interesting connections [3] with the non-commutative geometry developed by Connes [8].

2. About non-differentiable functions: definitions and notations

In the following, we consider continuous real-valued functions $x(t)$, defined on a compact set I of $\mathbf{R}$.

We denote by $\mathscr{C}^0(I)$ the set of continuous functions (denoted by $C^0(I)$) which are *nowhere differentiable* on I.

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[1] For more details about the history of fractal space-time, we refer to [10, Section 2], and to the book of Sidharth [18].

[2] El Naschie [9] tries to go even further, and gives up the continuity hypothesis in his concept of Cantorian fractal space-time.

[3] For connections of scale relativity with string theory and El Naschie's Cantorian $\mathscr{E}^{(\infty)}$ space-time, we refer to [7,11].

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Definition 1. Let $0 < \alpha < 1$ and $x(t) \in C^0(I)$. The α-right and α-left local fractional derivatives of x at point $t \in I$ are defined by

$$\frac{d_+^\alpha x}{dt} = \lim_{h \to 0+} \frac{x(t+h) - x(t)}{h^\alpha}, \qquad \frac{d_-^\alpha x}{dt} = \lim_{h \to 0+} \frac{x(t) - x(t-h)}{h^\alpha}. \tag{1}$$

We denote by $\mathscr{C}^\alpha(I)$ the set of functions $x(t) \in \mathscr{C}^0(I)$ such that $d_+^\alpha x/dt$ and $d_-^\alpha x/dt$ exist for all $t \in I$. We refer to [2,3] for more details.

We refer to the book of Tricot [19, p. 152] for a definition of the *fractal dimension* of a graph. An important property of $\mathscr{C}^\alpha(I)$ is the following:

Lemma 2. *Let $0 < \alpha < 1$. For all functions $x(t) \in \mathscr{C}^\alpha(I)$, the fractal dimension of the graph of $x(t)$, $t \in I$, is constant, and equal to $2 - \alpha$.*

This set is very special. Indeed, the order of left–right derivation does not change when $t \in I$ varies. We introduce a new functional space in order to allow a changing order of derivation.

Definition 3. Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a continuous function such that $0 < \alpha(t) < 1$ for all $t \in I$. We denote by $\mathscr{C}^{\alpha(t)}(I)$ the set of functions $x(t) \in \mathscr{C}^0(I)$ such that, for all $t \in I$, the $\alpha(t)$-right and -left derivatives exist.

This functional space will play a crucial role in special scale relativity. In particular, using Lemma 2, we can see that *locally*, the fractal dimension of a function belonging to $\mathscr{C}^{\alpha(t)}(I)$ for a given continuous function $\alpha(t)$ is more or less constant, but it can strongly vary along the path.

3. Galilean scale relativity

3.1. About the scale reference system

Let $\epsilon_0 > 0$ be a real number which, in the following, is a *resolution* variable. A given *absolute* resolution $\epsilon > 0$ can be described with respect to a given *origin* of resolution, ϵ_0, by the new variable

$$s_{\epsilon_0}(\epsilon) = \frac{\epsilon}{\epsilon_0}, \tag{2}$$

which is now a *scale*.

In order to obtain a "classical" reference system for scale, we introduce, for each $\epsilon_0 > 0$ fixed, the function

$$E_{\epsilon_0}(\epsilon) = \ln(\epsilon/\epsilon_0). \tag{3}$$

In this scale reference system, we have ϵ_0 which is sent by E_{ϵ_0} to 0, and for resolutions ϵ such that $\epsilon > \epsilon_0$ (resp. $\epsilon < \epsilon_0$), we have $E_{\epsilon_0}(\epsilon) > 0$ (resp. $E_{\epsilon_0}(\epsilon) < 0$).

Definition 4. We denote by $\mathscr{R}_E^{\epsilon_0}$ the scale reference system, related to resolution via the function E_{ϵ_0} defined by $E_{\epsilon_0}(\epsilon) = \ln(\epsilon/\epsilon_0)$.

The scale reference system is less natural than the resolution reference system. However, in order to easily write the analogy between the relativity principle of Einstein and the scale relativity principle of Nottale, the scale reference system is more appropriate.

3.1.1. Change of origin in the scale reference system

We now study the effect of changing the origin of a scale reference system $\mathscr{R}_E^{\epsilon_0}$ from ϵ_0 to ϵ_1. We have

$$E_{\epsilon_1}(\epsilon) = \ln(\epsilon/\epsilon_1) = \ln((\epsilon/\epsilon_0)(\epsilon_0/\epsilon_1)) = E_{\epsilon_0}(\epsilon) + \ln(\epsilon_0/\epsilon_1). \tag{4}$$

The basic effect of changing origin of resolution is then a *translation* in the scale reference system.

The quantity $\ln(\epsilon_0/\epsilon_1)$ is the *scale speed* of the scale reference system $\mathscr{R}_E^{\epsilon_1}$ with respect to $\mathscr{R}_E^{\epsilon_0}$. This terminology is justified by the following "Galilean" composition rule of scale speed:

Lemma 5. *Let ϵ_1, ϵ_2 and $\epsilon > 0$ be three resolutions. The scale speed of $\mathscr{R}^{\epsilon_3}$ with respect to $\mathscr{R}^{\epsilon_1}$, denoted $S_{3/1}$, satisfies*

$$S_{3/1} = S_{3/2} + S_{2/1}, \tag{5}$$

where $S_{3/2}$ (resp. $S_{2/1}$) is the scale speed of $\mathscr{R}^{\epsilon_3}$ (resp. $\mathscr{R}^{\epsilon_2}$) with respect to R^{ϵ_2} (resp. $\mathscr{R}^{\epsilon_1}$).

Proof. We have

$$S_{3/1} = \ln(\epsilon_1/\epsilon_3) = \ln\left(\frac{\epsilon_1}{\epsilon_2}\frac{\epsilon_2}{\epsilon_3}\right) = \ln(\epsilon_1/\epsilon_2) + \ln(\epsilon_2/\epsilon_3) = S_{2/1} + S_{3/2}. \qquad \square$$

3.2. Construction of a reference system for non-differentiable one-dimensional manifolds

In this section, we discuss the construction of an *intrinsic* coordinates system for one-dimensional non-differentiable manifolds. The basic example is the graph of an everywhere non-differentiable *continuous* function.

3.2.1. Curvilinear coordinates and Lebesgue's theorem

Let $x(t)$ be a continuous differentiable function, defined on a compact set I of $\mathbf{R}$. The basic way to construct an intrinsic coordinates system on the graph Γ of $x(t)$ is to introduce the so-called *curvilinear* coordinate, which is defined, an origin $t_0 \in I$ being given, by the *length* $L(x; t, t_0)$ of the graph of $x(t)$ between the points $x(t_0)$ and $x(t)$.

If $x(t)$ is nowhere differentiable, one cannot use this construction. Indeed, we have the *converse* of Lebesgue's theorem:

Theorem 6. *If $x(t)$ is almost everywhere non-differentiable then the length of $x(t)$ is infinite.*

As a consequence, we cannot define the analogue of curvilinear coordinates on a nowhere differentiable curve. Following Nottale, we introduce a new point of view on this problem: in general, we can have access not to the nowhere differentiable function $x(t)$, but to a "representation" of it, controlled by the resolution constraint, and to the behaviour of this representation when the resolution changes. [4] In the following we study the one-parameter family of mean representation of $x(t)$ and study its properties.

3.2.2. Representation theory of real-valued functions

We introduce the general idea of representation of a given real-valued function. This notion comes from Nottale's original work on fractal functions [16].

Let $x(t)$ be a real-valued function, defined on a compact set I of $\mathbf{R}$ (or defined on $\mathbf{R}$).

Definition 7. A representation of $x(t)$ is a one-parameter family of real-valued functions, denoted $X(t;\epsilon)$, $\epsilon \in \mathbf{R}^+$, such that

1. for all $\epsilon \in \mathbf{R}^+$, the function $X(t;\epsilon)$ is differentiable;
2. we have simple convergence toward $x(t)$ when ϵ goes to zero, i.e., $X(t;\epsilon) \overset{\epsilon\to 0}{\to} x(t)$.

A basic example is to take the ϵ-mean function as a representation of x, i.e., $X(t;\epsilon) = (1/2\epsilon)\int_{t-\epsilon}^{t+\epsilon} x(s)\,\mathrm{d}s$. This is the representation that we use in the following. A general study of representation of real-valued functions is done in [6]. Following Nottale [16, p. 75], we define the converse notation of *fractal functions*:

Definition 8. A fractal function is a real-valued function $F(t;\epsilon)$, depending on a parameter $\epsilon > 0$, such that

1. $F(t;\epsilon)$ is differentiable (except at a finite number of points) for all $\epsilon > 0$;
2. there exists a non-differentiable continuous function $f(t)$ such that $\lim_{\epsilon\to 0} F(t;\epsilon) = f(t)$.

The main point in this definition is that, contrary to the representation of the continuous function, we only know that the limit $f(t)$ exists. This does not imply that $f(t)$ is explicit.

Representations correspond to fractal functions for which the limiting function is explicit. We denote by $\mathscr{F}$ the set of fractal functions. An interesting example of fractal functions is introduced by Nottale [16]:

[4] This can be considered as the beginning of the renormalization group approach.

Definition 9. For all $\epsilon > 0$ and $\mu > 0$, we denote by $\Phi_{\epsilon,\mu}(x, y)$ a continuous function such that

$$\int_{-\infty}^{\infty} \Phi_{\epsilon,\mu}(x, y)\,\mathrm{d}y = 1 \quad \forall x \in \mathbf{R}. \tag{6}$$

Such a function is called a smoothing function.

We call Nottale's functions fractal functions satisfying for all $\epsilon > 0$, $\forall 0 < \mu < \epsilon$,

$$x(t, \epsilon) = \int \Phi_{\epsilon,\mu}(t, y) x(y, \mu)\,\mathrm{d}y. \tag{7}$$

As a natural example, we can take mean functions by using the Dirac window $\Phi_\epsilon(x, y) = (1/2\epsilon)\mathbf{1}_{[x-\epsilon,x+\epsilon]}(y)$, where the function $\mathbf{1}_{\bullet}$ is defined for the whole interval $\bullet$ by: $\mathbf{1}_{\bullet}(y) = 1$ if $y \in \bullet$ and $\mathbf{1}_{\bullet}(y) = 0$ otherwise.

We denote by $\mathscr{N}(\Phi_{\epsilon,\mu})$ the set of Nottale functions satisfying (7) with a smoothing function equal to $\Phi_{\epsilon,\mu}$.

Remark 10. It is important to fix the smoothing function. If not, we have no interesting equivalence relation on this set (see Section 3.2.3).

3.2.3. About equivalence relations on fractal functions

We have a natural equivalence relation on $\mathscr{F}$, given by:

Definition 11. We say that two fractal functions, denoted $F_1(t; \epsilon)$ and $F_2(t; \epsilon)$, are equivalent, and we denote $F_1 \sim F_2$ if $\lim_{\epsilon\to 0} F_1(t; \epsilon) = \lim_{\epsilon\to 0} F_2(t; \epsilon)$.

We easily verify that $\sim$ is an equivalence relation. The basic idea behind the equivalence relation $\sim$ is that a given non-differentiable function admits an infinite number of fractal functions as representation.

Problem 12. Can we define a canonical representation of a non-differentiable function?

The mean representation used in Section 3.2.4 seems to be a good candidate.

The main problem of this relation is that it is based on the limiting function, which is not always accessible. To solve this problem, Nottale introduces the following binary relation:

Definition 13. Let F_1 and F_2 be in $\mathscr{F}$. We say that F_1 and F_2 are equivalent, and we denote $F_1 \mathscr{R} F_2$ if $\forall_\epsilon, \forall t$, we have $|F_1(t; \epsilon) - F_2(t; \epsilon)| < \epsilon$.

The problem is that $\mathscr{R}$ is not an equivalence relation on $\mathscr{F}$ because it does not respect the *transitivity property*, as proved by the following counter-example.

Let $F_1(t; \epsilon)$ be given. We define $F_2(t; \epsilon) = F_1(t; \epsilon) + (2/3)\epsilon$ and $F_3(t; \epsilon) = F_1(t; \epsilon) + (4/3)\epsilon$. We have $|F_1(t; \epsilon) - F_2(t; \epsilon)| = (2/3)\epsilon < \epsilon$ for all $\epsilon > 0$ and t. Moreover, we have $|F_2(t; \epsilon) - F_3(t; \epsilon)| = (2/3)\epsilon < \epsilon$ for all $\epsilon > 0$ and t. We deduce that $F_1 \mathscr{R} F_2$ and $F_2 \mathscr{R} F_3$. However, we have $|F_1(t; \epsilon) - F_3(t; \epsilon)| = (4/3)\epsilon > \epsilon$ for all $\epsilon > 0$ and t, such that $F_1 F_3$.

However, we have the following lemma:

Lemma 14. *For all smoothing functions $\Phi_{\epsilon,\mu}$ satisfying* (6), *the binary relation $\mathscr{R}$ is an equivalence relation on Nottale's set $\mathscr{N}(\Phi_{\epsilon,\mu})$.*

Proof. The only non-trivial part is the transitivity property. Let $f(x, \epsilon)$, $g(x, \epsilon)$ and $h(x, \epsilon)$ be three Nottale functions of $\mathscr{N}(\Phi_{\epsilon,\mu})$ such that $f \mathscr{R} g$ and $g \mathscr{R} h$. We have

$$|f(x, \epsilon) - h(x, \epsilon)| = \left| \int \Phi_{\epsilon,\mu}(t, y)(f(y, \mu) - h(y, \mu))\,\mathrm{d}y \right|. \tag{8}$$

As $f \mathscr{R} g$ and $g \mathscr{R} h$, we have, for all $x \in \mathbf{R}$ and for all $\mu > 0$,

$$|f(x, \mu) - h(x, \mu)| < |f(x, \mu) - g(x, \mu)| + |g(x, \mu) - h(x, \mu)| < 2\mu. \tag{9}$$

Hence, we obtain for all $x \in \mathbf{R}$ and $\forall \epsilon > 0$ and $0 < \mu < \epsilon$,

$$|f(x,\epsilon) - h(x,\epsilon)| < 2\mu \int \Phi_{\epsilon,\mu}(x,y)\,\mathrm{d}y = 2\mu. \tag{10}$$

By choosing $\mu = \epsilon/2$, we obtain $f\mathscr{R}h$. This concludes the proof. □

Remark 15. A notion of the scale equivalence relation is defined in Section 3.2.4.

3.2.4. One-parameter family of mean functions and reference system

The idea, which is taken from Nottale's work, is to associate to $x(t)$ the one-parameter family of mean functions $x_\epsilon(t)$ defined for all $\epsilon > 0$ by

$$x_\epsilon(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} x(t)\,\mathrm{d}t. \tag{11}$$

For all $\epsilon > 0$, the mean function $x_\epsilon(t)$ is differentiable, with a derivative equal to $(\mathrm{d}x_\epsilon/\mathrm{d}t)(t) = (x(t+\epsilon) - x(t-\epsilon))/2\epsilon$. As a consequence, for all $\epsilon > 0$, a $t_0 \in I$ being fixed, we can define an intrinsic coordinates system, which is simply the curvilinear coordinate associated to $x_\epsilon(t)$, denoted $X_\epsilon(t)$ and defined by

$$X_\epsilon(t) = L(x_\epsilon; t, t_0). \tag{12}$$

We are led to the following natural reference system associated to $x(t)$:

Definition 16. Let $t_0 \in I$ be fixed. For all $\epsilon > 0$, the $(E(\epsilon), t_0)$-space reference system, denoted by $R_X^{(E,t_0)}$, is defined by $X_\epsilon(t)$.

As ϵ can be recovered by $E_{\epsilon_0}(\epsilon)$, we will now denote X_ϵ by X_E. We remark that, by definition of X_E, we have

$$\frac{\mathrm{d}X_E}{\mathrm{d}t} = \left|\frac{x(t+\epsilon) - x(t-\epsilon)}{2\epsilon}\right|, \tag{13}$$

so that we recover the classical *mean velocity*.

The approach of non-differentiable functions via the one-parameter family of mean functions induces a natural *equivalence relation in scale* for continuous functions.

Definition 17. Let $f(t)$ and $g(t)$ be two continuous real-valued functions defined on a compact set I of $\mathbf{R}$. For all $\epsilon > 0$, we denote by $f_\epsilon(t)$ and $g_\epsilon(t)$ the ϵ-mean function of f and g, respectively. We say that f and g are ϵ-scale equivalent, and we denote $f \sim_\epsilon g$ if $f_\epsilon(t) = g_\epsilon(t)$ for all $t \in I$.

One can easily verify that $\sim_\epsilon$ is an equivalence relation.

For a differentiable function f, this equivalence relation is not interesting, because the *asymptotic* object, f, can be studied via ordinary differential calculus. If f is a non-differentiable function, a new phenomenon appears. Indeed, there exists a *non-zero minimal resolution* (see Section 3.5), denoted $\epsilon(f)$, under which one must take into account the non-differentiability of f. In that case, for a given ϵ and a given function f, we have an *infinite*-dimensional equivalence class of functions, g, which cannot be distinguished by f at resolution $\epsilon(f)$. This is why one must deal with an infinite number of *representations* of f when a minimal resolution exists [5] (which is the case in scale relativity).

3.2.5. Relation between the space reference system and scale: the scale law

The relation between the space reference system and scales can be described by a *scale law*, which is an ordinary differential equation controlling the behaviour of $X_\epsilon(t)$ when ϵ changes.

Definition 18. We say that $x(t)$ satisfy a scale law if there exists a function $A : \mathbf{R} \to \mathbf{R}$ such that for all $\epsilon > 0$ we have

$$\frac{\mathrm{d}X_E}{\mathrm{d}E} = A(X_E), \tag{14}$$

where E is the scale variable, an origin of resolution $\epsilon_0 > 0$ being given.

[5] This phenomenon is responsible for the existence of an infinity of geodesics in scale relativity.

In [4], we prove the following lemma:

Lemma 19. *Let* $0 < \alpha < 1$ *and* $x \in \mathscr{C}^{\alpha}(I)$. *For all* $\epsilon \geqslant 0$ *sufficiently small, we denote by* $r_+(t,\epsilon)$ (*resp.* $r_-(t,\epsilon)$) *the remainder of the generalized Taylor expansion of* $f(t+\epsilon)$ (*resp.* $f(t-\epsilon)$), *i.e.*,

$$x(t+\sigma\epsilon) = x(t) + \sigma\epsilon^{\alpha}\frac{\mathrm{d}_{\sigma}^{\alpha}x}{\mathrm{d}t} + r_{\sigma}(t,\epsilon), \quad \sigma = \pm, \tag{15}$$

and $r(t,\epsilon) = r_+(t,\epsilon) - r_-(t,\epsilon)$.

If $r(t,\epsilon)$ *is differentiable on an open neighbourhood of* 0 *with respect to* ϵ, *then* $x(t)$ *satisfies the scale law*

$$A(x) = (\alpha - 1)x + \mathrm{O}(x^2). \tag{16}$$

We refer to [4] for a proof.

This result motivates the introduction of a new functional space:

Definition 20. We denote by $\mathscr{C}_L^{\alpha}(I)$ the subset of $\mathscr{C}^{\alpha}(I)$ whose functions satisfy the linear scale law

$$\frac{\mathrm{d}X_E}{\mathrm{d}E} = (\alpha - 1)X_E. \tag{17}$$

Galilean relativity is based on functions belonging to $\mathscr{C}_L^{\alpha}(I)$.

3.3. Djinn variable

For $\epsilon_0 > 0$ fixed, and for all $\epsilon > 0$ being given, we have defined an intrinsic coordinate on the graph of a non-differentiable function $x(t) \in \mathscr{C}_L^{\alpha}(I)$ by taking X_{ϵ}, an origin $t_0 \in I$ being given.

We introduce a new variable $\mathscr{X}_E$ which is defined by

$$\mathscr{X}_E = \ln X_E \quad \text{if } X_E > 0. \tag{18}$$

Remark 21. We refer to [16, p. 218] for a possible justification of the logarithmic variable form.

In this new variable, the scale law (17) reduces to

$$\frac{\mathrm{d}\mathscr{X}_E}{\mathrm{d}E} = \alpha - 1. \tag{19}$$

We stress that the assumption $X_E > 0$ is nothing else than saying that only the behaviour for $t > t_0$ can be described, which means that the phenomenon is strongly *non-reversible*.

We denote by δ the parameter

$$\delta = 1 - \alpha \tag{20}$$

in the following. The parameter δ is called the *djinn* variable by Nottale.

3.4. The two basic effects: translation in scale and space

We now investigate the two basic effects of translating origin of scale and origin of space.

3.4.1. Translation in scale

As we have see in Section 3.1.1, a change in the origin of resolution between ϵ_0 and ϵ_1 translates for scale reference system in a translation given by

$$E_{\epsilon_1} = E_{\epsilon_0} + S(\epsilon_1, \epsilon_0), \tag{21}$$

where the *scale state* $S(\epsilon_1, \epsilon_0)$ is given by

$$S(\epsilon_1, \epsilon_0) = \ln(\epsilon_0/\epsilon_1). \tag{22}$$

This translation in scale is viewed in the space reference system using the scale law (17) by integrating the ordinary differential equation between E_{ϵ_0} and E_{ϵ_1}. We obtain

$$X_{E_{\epsilon_1}} = X_{E_{\epsilon_0}} \exp(-\delta S(\epsilon_1, \epsilon_0)). \tag{23}$$

For the variable $\mathscr{X}_E$, translation in scale gives

$$\mathscr{X}_{E_{\epsilon_1}} = \mathscr{X}_{E_{\epsilon_0}} - \delta S(\epsilon_1, \epsilon_0), \tag{24}$$

which is, following Nottale, the *Galilean* version of the scale relativity theory.

We stress that, in this case, the *djinn* variable is constant under scale translations, which is of course, a very particular case, i.e.,

$$\delta_{E_{\epsilon_1}} = \delta_{E_{\epsilon_0}}, \tag{25}$$

or

$$\frac{d\delta}{dE} = 0. \tag{26}$$

The djinn variable plays the same role as the time t in classical Galilean relativity theory.

3.4.2. Translation in space

Assume that we make a change of origin in the space reference system, by changing t_0 to t_1. We have the following relation, the origin of resolution ϵ_0 begin fixed:

$$X_E^{t_1} = X_E^{t_0} + T(t_0, t_1; E), \tag{27}$$

where the *space state* is defined by

$$T(t_0, t_1; E) = \int_{t_1}^{t_0} x_E(s)\, ds. \tag{28}$$

The translation depends on E. If we assume that T is independent of E (which is the case in Nottale's papers), a minimal resolution appears (see the remark below). Here, we prove that such an effect does not exist. Indeed, by differentiating Eq. (27) as

$$\frac{dT}{dE} = -\delta T, \tag{29}$$

we obtain

$$\frac{dX_E^{t_1}}{dE} = \frac{dX_E^{t_0}}{dt} - \delta T = -\delta X_E^{t_0} - \delta T = -\delta X_E^{t_1} + \delta T - \delta T = -\delta X_E^{t_1}, \tag{30}$$

which is the same equation as for $X_E^{t_0}$.

Remark 22. If we assume for simplicity that the space state is a function independent of E, we have

$$X_E^{t_1} = X_E^{t_0} + T(t_0, t_1). \tag{31}$$

We obtain

$$\frac{dX_E^{t_1}}{dE} = -\delta X_E^{t_1} + \delta T(t_0, t_1), \tag{32}$$

which gives

$$\frac{d\mathscr{X}_E^{t_1}}{dE} = -\delta + \delta \exp(\mathscr{X}_E^{t_1}). \tag{33}$$

As a consequence, we have

$$X_{E_{\epsilon_1}}^{t_1} = T[1 + S(\lambda, \epsilon_1)^{\delta}], \tag{34}$$

where λ is a resolution defined by

$$\lambda = \left(\frac{X^{t_0}_{E_{\epsilon_0}}}{T}\right)^{1/\delta} \frac{1}{\epsilon_0}. \tag{35}$$

As explained by Nottale [14], the resolution λ has a particular status. Indeed, for $\epsilon \gg \lambda$, we have $X^{t_1}_{E_{\epsilon_1}} \sim T$, which is a typical differentiable behaviour. In contrast, for $\epsilon \ll \lambda$, we must take into account $S(\lambda, \epsilon_1)^{\delta}$, which comes from the non-differentiable character of $x(t)$. By definition λ is a relative resolution (it depends on ϵ_0) and is not at all scale invariant, being dependent on x via $X^{t_0}_{E_{\epsilon_0}}$.

We define in the next section a natural notion of minimal resolution, which is compatible with the result of this section.

3.5. Minimal resolution

The domain of validity of scale relativity is mainly beyond classical mechanics, in particular, particle physics and quantum mechanics. A common idea, even at the basis of *superstring theory* (see [12]), is that there exists a scale at which we must take into account the non-differentiable character of space-time. In the following, we define a natural notion of *minimal resolution* for a given non-differentiable function $x(t)$, denoted $\epsilon(x)$, such that for $\epsilon > \epsilon(x)$, the non-differentiable character of x is not dominant, and for $\epsilon < \epsilon(x)$, we must take into account non-differentiable effects.

3.5.1. ϵ-differentiability

Let $x(t)$ be a continuous real-valued function defined on an open set I of $\mathbf{R}$. We call ϵ-oscillation of x the quantity

$$\text{osc}_\epsilon x(t) = \sup\{x(t') - x(t''),\ t', t'' \in [t-\epsilon, t+\epsilon]\}.$$

We denote

$$\Lambda^\alpha x(t) = \sum_{s,s' \in [t-\epsilon, t+\epsilon]} \frac{|x(x) - x(s')|}{|s-s'|^\alpha}.$$

Let $x(t)$ be a continuous function on I such that $\Lambda^\alpha x(t) \neq 0$ for all $t \in I$. We denote

$$a_{\epsilon,\alpha} x(t) = \frac{\text{osc}_\epsilon x(t)}{2\epsilon \Lambda^\alpha x(t)}. \tag{36}$$

For a differentiable function, we have $a_{\epsilon,1}x(t) \leqslant 1$ for all $t \in I$ and all ϵ. We introduce the following notion of ϵ-differentiability:

Definition 23. Let x be a continuous real-valued function defined on I. We assume that there exists $\alpha > 0$ such that $\Lambda^\alpha x(t) \neq 0$ for all $t \in I$. Let $\epsilon > 0$ be given. We say that x is ϵ-differentiable if $a_{\epsilon,\alpha}x(t) \leqslant 1$ for all $t \in I$.

We denote by $\epsilon(x)$ the minimal order of ϵ-differentiability:

$$\epsilon(x) = \inf\{\epsilon \geqslant 0,\ x \text{ is } \epsilon\text{-differentiable}\}. \tag{37}$$

We remark that for all $\lambda \in \mathbf{R}$ we have the following stability results:

$$\epsilon(\lambda x) = \epsilon(x), \qquad \epsilon(x+\lambda) = \epsilon(x).$$

3.5.2. Minimal resolution

The basic idea is that oscillation of non-differentiable functions increases toward infinity. In particular, *fractal* functions, according to Tricot [19], are precisely functions such that $\lim_{\epsilon \to 0} \text{osc}_\epsilon x(t)/\epsilon = \infty$ uniformly in t. We easily deduce that fractal functions possess a non-zero minimal order of ϵ-differentiability. This result is in fact general for non-differentiable functions:

Lemma 24. *Let $x(t)$ be a continuous, non-differentiable real-valued function. Then, its minimal order of ϵ-differentiability is non-zero.*

We refer to [4] for a proof.

In the following, we call $\epsilon(x)$ the minimal resolution of x. The minimal resolution is a pure geometric constant associated to the regularity of x.

Remark 25. Minimal resolution allows us to define the notion of quantum derivatives and scale derivative in [5].

3.6. Summary about Galilean scale relativity

We summarize the previous results in the following:

- Let $\epsilon_0 > 0$ and $\epsilon_1 > 0$ be two given resolutions. We denote by E and E' the scale variables with respect to ϵ_0 and ϵ_1, respectively.
- For all $\epsilon > 0$, we denote by X the associated space coordinate on Γ_ϵ.
- Let T be a given translation of origin in the space reference system. We denote by $Y = X - T$ the normalized coordinate on Γ_ϵ.
- Let $\mathscr{Y} = \ln Y$, $Y > 0$, and S be a translation of origin in the scale reference system. Then we have

$$\frac{\partial \mathscr{Y}}{\partial E} = -\delta, \qquad \frac{\partial \mathscr{Y}}{\partial \delta} = -S. \tag{38}$$

  Note that we have taken δ as a fundamental variable by obtaining S via a differentiation of Y with respect to δ, just as the classical speed is obtained as a derivative of the space variable with respect to time.
- If $S_{1/0}$ and $S_{2/1}$ are the scale speeds of the scale reference system 1 with respect to 0 and 2 with respect to 1, then the scale speed of the scale reference system 2 with respect to 0, denoted $S_{2/0}$, is given by the *Galilean* composition rule

$$S_{2/0} = S_{2/1} + S_{1/0}. \tag{39}$$

4. Special pure scale relativity

The existence of a minimal resolution $\epsilon(x)$ allows us to fix a specific origin of resolution by setting $\epsilon_0 = \epsilon(x)$. We keep notations from the previous section.

4.1. The Planck length as a scale invariant

The basic idea of Nottale is to generalize the previous Galilean point of view by allowing more general transformation laws of coordinates, respecting the relativity principle. In order to simplify the discussion, we restrict ourselves to a *pure scale* relativity theory, i.e., we only pay attention to the set of variables (Y, δ). We will discuss the general form of the special scale relativity (integrating the *time* variable) in a forthcoming paper.

If δ is not taken as an absolute variable, we know that the most general transformation rules of the form

$$\mathscr{Y}' = A(S)\mathscr{Y} + B(S)\delta, \qquad \delta' = C(S)\mathscr{Y} + D(S)\delta, \tag{40}$$

which respect the relativity principle are given by *Lorentz transformations* [13], i.e.,

$$\mathscr{Y}' = \frac{\mathscr{Y} - S\delta}{\sqrt{1 - (S^2/\mathscr{L}^2)}}, \qquad \delta' = \frac{\delta - S(\mathscr{Y}/\mathscr{L}^2)}{\sqrt{1 - (S^2/\mathscr{L}^2)}}, \tag{41}$$

where $\mathscr{L}$ is a constant.

Remark 26. The fact that δ is variable implies that the fractal dimension of quantum mechanical paths is fluctuating. As a consequence, by Lemma 2, we must consider continuous functions with a variable Hölder exponent, as, for example, the Riemann function

$$R(t)\sum_{n=1}^{\infty} \frac{\cos(n^2 t)}{n^2}. \tag{42}$$

In order to understand $\mathscr{L}$, we separate the geometric contribution of x and the "universal" one, by posing

$$\mathscr{L} = \ln\left(\frac{\epsilon(x)}{\Lambda}\right), \tag{43}$$

where Λ is a constant.

We have the following relation:

$$\log\left(\frac{\epsilon(x)}{\epsilon'}\right) = \frac{\log(\epsilon(x)/\epsilon) + \log S}{1 + \log(\epsilon(x)/\epsilon)\log S/\log^2(\epsilon(x)/\Lambda)}. \tag{44}$$

We denote by $\mathscr{T}_S$ the mapping giving ϵ' as a function of ϵ following Eq. (44), i.e., $\epsilon' = \mathscr{T}_S(\epsilon)$. The following lemma makes precise the status of this constant:

Lemma 27. *The constant Λ is scale invariant, i.e., $\mathscr{T}_S(\Lambda) = (\Lambda)$ for all $S \in \mathbf{R}$.*

Proof. We put $\epsilon = \Lambda$ in Eq. (44). We obtain

$$\frac{\log(\epsilon(x)/\Lambda) + \log S}{1 + \log(\epsilon(x)/\Lambda)\log S/\log^2(\epsilon(x)/\Lambda)} = \log(\epsilon(x)/\Lambda),$$

which concludes the proof. □

The minimal resolution is a geometrical constant, depending on the regularity of x (i.e., on the regularity of the space-time). The constant Λ is a universal constant, not depending on the geometry of the space-time manifold.

Remark 28. In [16, p. 94–5], the minimal resolution is identified with the De Broglie length, using the result of Abbott and Wise [1] on the Hausdorff dimension of a quantum mechanical path. The universal constant Λ is identified with the Planck length $\Lambda = \sqrt{\hbar G/c^3}$, where $\hbar$ is the reduced Planck constant, G is the gravitational constant, and c the speed of light (see [16, p. 235]).

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