STOCHASTIC PERTURBATION OF THE TWO-BODY PROBLEM

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Abstract. We study the impact of a stochastic perturbation on the classical two-body problem in particular concerning the preservation of first integrals and the Hamiltonian structure. Numerical simulations are performed which illustrate the dynamical behavior of the osculating elements as the semi-major axis, the eccentricity and the pericenter. We also derive a stochastic version of Gauss's equations in the planar case.

Keywords: Two-body problem, stochastic perturbation, numerical simulations, stochastic Gauss's equations

1 Introduction

The stability of the solar system is a famous open problem in celestial mechanics (see J. Moser (1978),S. Marmi (1999),J. Laskar (2010),J. Féjoz (2013a)). Since the discovery by Newton of the gravitation law, a mathematical approach to this problem is to study the stability of the n-body problem as an ideal model for the behavior of planetary systems. Numerous advances have been made in this direction. From the analytical point of view, recent contributions deal with application of the Kolmogorov-Arnold-Moser (KAM) theorem and Nekhoroschev theory to the n-body problem (see J. Féjoz (2004),M. Herman (1998),L. Niederman (1996)) for stability and Arnold's diffusion (J. Xia (1993),J. Xia (1994),J. Féjoz (2013b)) for instability. On the numerical side, simulations over very large time scales of the n-body problem in particular by S. Tremaine and Wisdom and J. Laskar give insight that the "solar system" is unstable (chaotic) over very large time.

A common feature of all these works is to deal with a deterministic model for the planetary motion. However, as pointed out by D. Mumford (1999), the meaning of such an assumption with respect to the real behavior of planetary systems is far from being satisfying. D. Mumford remarks that "a major step in making the equation more relevant is to add a small stochastic term".

However, adding a stochastic term is far from being trivial because the nature and origin of such a contribution is more or less unknown. A significant step has been done in Sharma & Parthasarathy (2007) (see also J. Cresson (2011)). Using observations made by I. Mann et al. (2004) about the stochastic fluctuations of the density of the zodiacal dust around the sun, Sharma & Parthasarathy (2007) propose a stochastic perturbation induced by a cloud whose density fluctuates stochastically.
in the whole text In this paper, we continue the study initiated by S.N. Sharma and H. Parthasarathy in several directions : first, we make a new derivation of the stochastic perturbation induced by a cloud having a stochastically fluctuating density. We then study how the classical properties of the two-body problem are affected by this stochastic perturbation. In particular, we discuss the persistence of first integrals like energy and angular momentum and the behavior of the Hamiltonian structure in the context of stochastic Hamiltonian systems introduced by Bismut (1981). Third, we perform numerical simulations in order to observe the dynamical behavior of the osculating elements. The accuracy of the numerical integrator is also discussed. Finally, we derive the stochastic version of Gauss's equations for the variations of the osculating elements which allows us to determine the contribution of the stochastic terms in the observed dynamical behavior.

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2 The two-body problem

We follow the book (Goldstein (2002),Chap.3) to which we refer for more details. In the whole text the symbol $^\mathsf{T}$ denotes the transpose of a vector.

2.1 Description

Let S and P be two bodies and M_S and M_P their masses. The body S is supposed to be the central body typically a star and P is the orbiting body typically a planet or a satellite. The motion is supposed to be in an elliptic configuration. The reduced mass is $m = \frac{M_S M_P}{M_S + M_P}$ and the potential coefficient is $k = GM_S M_P$ where G is the gravitational constant. We define $(S, \vec{x}, \vec{y})$ to be a fixed frame attached to S and $\vec{r}$ the position vector of P in this reference frame with ϕ his position angle. The elliptical motion is described with the semi-major axis a, the eccentricity e and the pericenter angle ω. We associate the polar reference frame $(S, \vec{e_R}, \vec{e_T})$ where $\vec{e_R}^\mathsf{T} = (\cos\phi, \sin\phi)$ and $\vec{e_T}^\mathsf{T} = (-\sin\phi, \cos\phi)$. In this reference frame we have $\vec{r} = r\vec{e_R}$ where r is the norm of the position vector. The motion is illustrated in Fig. 1
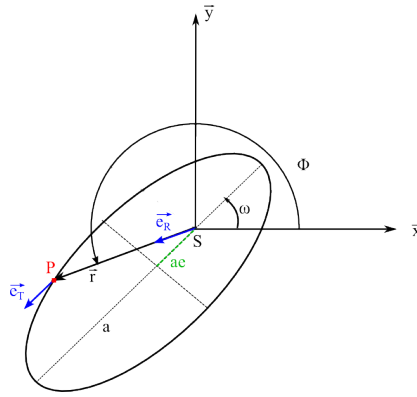

Fig. 1. The classical two body problem.

2.2 Equations of motion

The equations of motion for the two-body problem are given by

$$\begin{cases} \frac{dr}{dt} &= v, \\ \frac{d\phi}{dt} &= w, \\ \frac{dv}{dt} &= rw^2 - \frac{k}{mr^2}, \\ \frac{dw}{dt} &= -\frac{2vw}{r}. \end{cases} \tag{2.1}$$

Classical conserved quantities of motion are the *angular momentum* and *energy* of the system defined by

$$M = mr^2 w, \tag{2.2}$$

$$H = \frac{1}{2}m(v^2 + r^2 w^2) - \frac{k}{r}. \tag{2.3}$$

We will use also the *Laplace-Runge-Lenz vector* defined by $\vec{A} = m\vec{v} \wedge \vec{L} - km\vec{e_R}$ where $\vec{v}$ is the velocity vector and $\vec{L} = m\vec{r} \wedge \vec{v}$ is the angular momentum vector.

3 Perturbation induced by a dust-cloud

In Sharma & Parthasarathy (2007) the authors consider a stochastic perturbation induced by a cloud with a density which fluctuates stochastically. This assumption is supported by observations made by I. Mann et al. (2004) about the zodiacal dust around the Sun. These fluctuations comes from comets and asteroids which produce dust when they come near the Sun due to collisional fragmentation and sublimation, radiation pressure
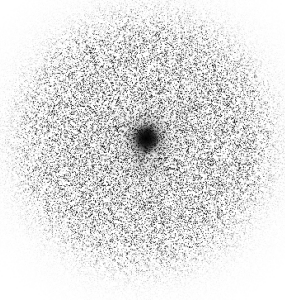
Fig. 2. The dust sphere.

acceleration and rotational bursting. In order to simplify the computations, we assume in the following that the dust cloud is a sphere.

The force $\vec{F}$ induced by a dust sphere of constant density ρ is only a radial force $\vec{F}^{\mathsf{T}} = (\frac{4}{3}\pi G\rho r, 0)$ (see Goldstein (2002) p.122). Our main assumption is that the mean density of this dust cloud is fluctuating randomly that is to say the density is a function of time define as

$$\rho(t) = \sigma_r W_t^r \tag{3.1}$$

where σ_r is a constant and W_t^r is a "white noise". The random force takes then the form

$$\vec{F}^{\mathsf{T}} = (mr\sigma_r W_t^r, 0) \tag{3.2}$$

Now if we add an isotropic tangential component induced by other physical process we obtain

$$\vec{F}^{\mathsf{T}} = \left(mr\sigma_r W_t^r, m\sigma_\phi W_t^\phi \right) \tag{3.3}$$

where σ_ϕ is a constant and W_t^ϕ is also a "white noise" independent of W_t^r.

4 Stochastic perturbation of the two-body problem

4.1 Reminder about stochastic differential equations

We remind basic properties and definition of stochastic differential equations in the sense of Itô. We refer to the book Øksendal (2003) for more details.

A *stochastic differential equation* is formally written (see Øksendal (2003),Chap.V) in differential form as

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t, \tag{4.1}$$

which corresponds to the stochastic integral equation

$$X_t = X_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s, \tag{4.2}$$

where the second integral is an Itô integral (see Øksendal (2003),Chap.III) and B_t is the classical Brownian motion (see Øksendal (2003),Chap.II,p.7-8).

An important tool to study solutions to stochastic differential equations is the *multi-dimensional Itô formula* (see Øksendal (2003),Chap.III,Theorem 4.6) which is stated as follows :

We denote a vector of Itô processes by $\mathbf{X}_t^\mathsf{T} = (X_{t,1}, X_{t,2}, \ldots, X_{t,n})$ and we put $\mathbf{B}_t^\mathsf{T} = (B_{t,1}, B_{t,2}, \ldots, B_{t,n})$, $d\mathbf{B}_t^\mathsf{T} = (dB_{t,1}, dB_{t,2}, \ldots, dB_{t,n})$. We consider the multi-dimensional stochastic differential equation defined by (4.1). Let f be a $\mathcal{C}^2(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$-function and X_t a solution of the stochastic differential equation (4.1). We have

$$df(t, \mathbf{X}_t) = \frac{\partial f}{\partial t} dt + (\nabla_{\mathbf{X}}^\mathsf{T} f) d\mathbf{X}_t + \frac{1}{2}(d\mathbf{X}_t^\mathsf{T})(\nabla_{\mathbf{X}}^2 f) d\mathbf{X}_t, \tag{4.3}$$

where $\nabla_{\mathbf{X}} f = \partial f/\partial \mathbf{X}$ is the gradient of f w.r.t. X, $\nabla_{\mathbf{X}}^2 f = \nabla_{\mathbf{X}} \nabla_{\mathbf{X}}^\mathsf{T} f$ is the Hessian matrix of f w.r.t. $\mathbf{X}$, δ is the Kronecker symbol and the following rules of computation are used : $dt dt = 0$, $dt dB_{t,i} = 0$, $dB_{t,i} dB_{t,j} = \delta_{ij} dt$.

4.2   The stochastic two-body problem

The general form of the equations of the perturbed two-body problem by a planar force $\vec{F} = (F_r, F_\phi)$ is easily compute and reads

$$\begin{cases} \frac{dr}{dt} &= v, \\ \frac{d\phi}{dt} &= w, \\ \frac{dv}{dt} &= rw^2 - \frac{k}{mr^2} + \frac{F_r}{m}, \\ \frac{dw}{dt} &= -\frac{2vw}{r} + \frac{F_\phi}{mr}, \end{cases} \tag{4.4}$$

which gives, replacing F by the random force (3.3) :

$$\begin{cases} \frac{dr}{dt} &= v, \\ \frac{d\phi}{dt} &= w, \\ \frac{dv}{dt} &= rw^2 - \frac{k}{mr^2} + r\sigma_r W_t^r, \\ \frac{dw}{dt} &= -\frac{2vw}{r} + \frac{\sigma_\phi W_t^\phi}{r}. \end{cases} \tag{4.5}$$

The classical way to give a sense to this set of equations is to replace W_t by a suitable stochastic process called the *white noise process* which is heuristically obtained as increment of the *Brownian motion* B_t (see Øksendal (2003),p.7-8). We then obtain a stochastic differential equation in Itô sense given by

$$\begin{cases} dr &= v dt, \\ d\phi &= w dt, \\ dv &= \left(rw^2 - \frac{k}{mr^2}\right) dt + r\sigma_r dB_t^r, \\ dw &= -\frac{2vw}{r} dt + \frac{\sigma_\phi}{r} dB_t^\phi, \end{cases} \tag{4.6}$$

where B_t^r and B_t^ϕ are independent. This set of equations describes what we called the *stochastic two-body problem* in the following.

5   Symmetries and First integrals

First integrals and symmetries play a fundamental role in classical mechanics and in particular for the study of the deterministic n-body problem (see V.I. Arnold (1989)). A natural question is to know if symmetries and first integrals of a given deterministic system persist in an appropriate sense. In this Section, we remind the definition of weak and strong first integrals. We prove that the angular momentum is preserved under stochastic perturbation and give rise to a weak first integral of the stochastic two-body problem.

5.1   Definitions

Let $dx/dt = f(x,t)$, $x \in \mathbb{R}^n$ $(\star)$ be an ordinary differential equation. A function $I : \mathbb{R}^n \mapsto \mathbb{R}$ is called a *first integral* of $(\star)$ if for all solutions x_t of $(\star)$ we have $I(x_t) = I(x_0)$ for all t. If I is sufficiently smooth we deduce $\frac{dI(x_t)}{dt} = 0$.

A natural generalisation of this definition in the setting of stochastic differential equations is given for example in Misawa (1999) (see also Bismut (1981),Cresson-Darses (2007a),Cresson-Darses (2007b) and J.A. Lázaro Camí (2008),p.52):

Strong first integral A function $I : \mathbb{R}^n \to \mathbb{R}$ is a *strong first integral* of (4.1) if for all solutions X_t of (4.1), the stochastic process $I(X_t)$ is a constant process, i.e. $I(X_t) = I(X_0)$ or $d(I(X_t)) = 0$.

Such a property is very strong and classical first integral are usually not preserved in the strong sense. However, a weaker property can be looked for:

Weak stochastic first integral A function $I : \mathbb{R}^n \to \mathbb{R}$ is a weak stochastic first integral of (4.1) if for all solutions X_t of (4.1), the stochastic process $I(X_t)$ satisfies $\mathrm{E}(I(X_t)) = \mathrm{E}(I(X_0))$ where E denotes the expectation.

This condition implies that $I(X_t) = I(X_0)$ almost surely.

5.2 *Variation of the angular momentum and the energy*

Using formulas (2.2) and (2.3) for the angular momentum and energy, the multi-dimensional Itô formula with $X_t^\mathsf{T} = (r, \phi, v, w)$ and $B_t^\mathsf{T} = \left(B_t^r, B_t^\phi \right)$ leads to

$$dM(X_t) = mr\sigma_\phi dB_t^\phi, \quad dH(X_t) = mrv\sigma_r dB_t^r + mrw\sigma_\phi dB_t^\phi + \frac{m}{2} \left[\sigma_r^2 r^2 + \sigma_\phi^2 \right] dt.$$

for the behavior of these first integrals over solutions of the stochastic two-body problem. As expected, there is no persistence of the angular momentum or energy integral in the strong sense.

Remark The strong conservation of the angular momentum is broken by our assumption that an isotropic tangential force exists, i.e. $\sigma_\phi \neq 0$ (see §.3, equation (3.3)).

However, we have the following weak conservation property :

Lemma 5.1 *The angular momentum is a weak first integral of the stochastic two-body problem.*

The proof is simple and relies on classical properties of the Brownian motion.

Proof Let X_t be a solution of the stochastic two-body problem. We have $M(X_t) = M(X_0) + \int_0^t mr\sigma_\phi dB_t^\phi$ where M is the angular momentum function. Using the property that $\mathrm{E}\left(\int_a^b f dB \right) = 0$ for all f sufficiently smooth (see Øksendal (2003),Definition 3.4,p.18 and Theorem 3.7 (iii),p.22), we deduce that $\mathrm{E}(M(X_t)) = \mathrm{E}(M(X_0))$ which concludes the proof.

This result does not extend to the energy first integral. This is due to the existence of a non-trivial deterministic term emerging in the Itô formula. Precisely, we have $H(X_t) = H(X_0) + \int_0^t mrv\sigma_r dB_t^r + \int_0^t mrw\sigma_\phi dB_t^\phi + \frac{m}{2} \int_0^t \left[\sigma_r^2 r^2 + \sigma_\phi^2 \right] ds$. Taking expectation, we obtain

$$\mathrm{E}(H(X_t)) = \mathrm{E}(H(X_0)) + \frac{m}{2} \mathrm{E}\left(\int_0^t \left[\sigma_r^2 r^2 + \sigma_\phi^2 \right] ds \right).$$

The second term is non zero so that the energy first integral is not preserved even in a weak sense.

The conservation of the angular momentum in the weak sense will be an important information in order to perform simulations because it will be the only quantity that we could check his conservation during the simulations.

6   Simulations

The simulation of stochastic differential equations is more difficult than in the deterministic case (see Kloeden et al. (1999) and Higham (2001)). In the sequel, we use a stochastic Runge-Kutta of weak order 2 due to N.J. Kasdin and L.J. Stankievech in Kasdin et al. (2009). The term of *weak order* refers to the error of the stochastic numerical scheme with respect to the expectation of the solution computed.

Our simulations are made with the same initial conditions and integration time used by Sharma & Parthasarathy (2007), which are :

$$
\begin{aligned}
r(0) &= 1 \ AU, & (6.1) \\
\phi(0) &= 1 \ rad, & (6.2) \\
v(0) &= 0.01 \ AU/TU, & (6.3) \\
\omega(0) &= 1.1 \ rad/TU, & (6.4) \\
\sigma_r &= 0.0121 \ TU^{-3/2}, & (6.5) \\
\sigma_\phi &= 2.2 \times 10^{-4} \ AU.TU^{-3/2}, & (6.6)
\end{aligned}
$$

where AU is the Astronomical Unit which is the Earth-Sun distance and TU is the Time Unit which is approximately 58 days. These units are called canonical units (see Bate et al. (1971)).

The initials conditions are chosen such that the unperturbed motion is an ellipse and the diffusion constants σ_r and σ_ϕ are chosen such that the stochastic perturbing force is proportional to 1/10 of the gravitational force at the initial time. Numerical integration are performed over 15TU like in Sharma & Parthasarathy (2007). The unperturbed trajectory as well as the perturbed one are plotted in Fig. 3 with color green and red respectively and we still use the same colors on figures to refer to the unperturbed and perturbed case. The accuracy
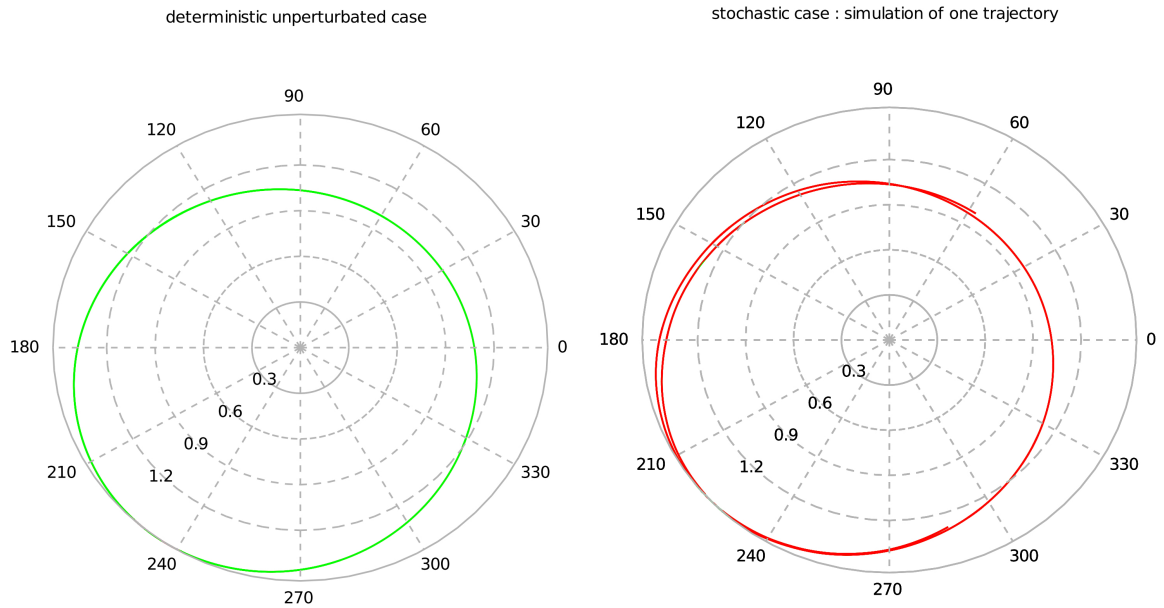

Fig. 3. Left: Unperturbed case. **Right:** Perturbed case.

of the integrator can be tested by looking for the preservation of the weak first integral given by the angular momentum. Expectations are computed using a Monte Carlo method. Our result indicates a very good behavior of the integrator with respect to weak first integrals (see Fig. 4). The dynamical behavior of the semi-major axis, the eccentricity and the pericenter angle for the trajectory of the perturbed motion corresponding to Fig. 3 is given in Fig. 5 as well as the expectation of these elements in Fig. 6.

7   Stochastic planar Gauss equations

To study the variations of orbital elements we derive a stochastic version of the classical Gauss equations (see Goldstein (2002),p.96-103). The strategy to obtain these equations in the planar case is as follows. We consider a stochastic perturbation per unit of mass in polar coordinates

$$
d\vec{v}_P = \begin{pmatrix} \bar{R} \\ \bar{T} \end{pmatrix} dt + (\tilde{R}, \tilde{T}) \cdot d\mathbf{B} \tag{7.1}
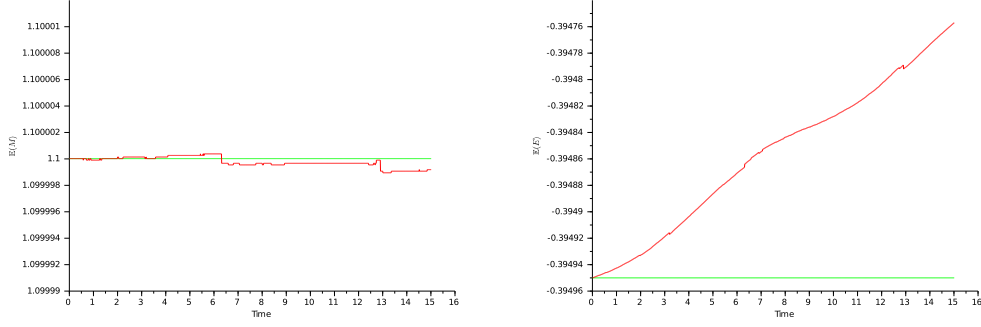$$

Fig. 4. Left: $\mathrm{E}(M(X_t))$. **Right:** $\mathrm{E}(H(X_t))$.
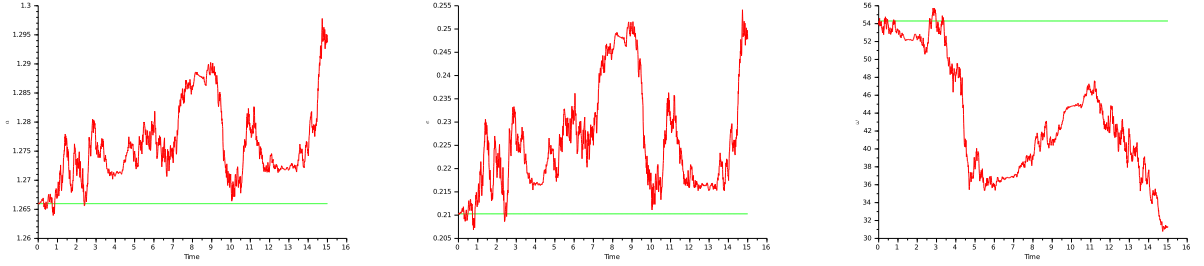
Fig. 5. Left: a the semi-major axis. **Center:** e the eccentricity. **Right:** ω the pericenter angle.
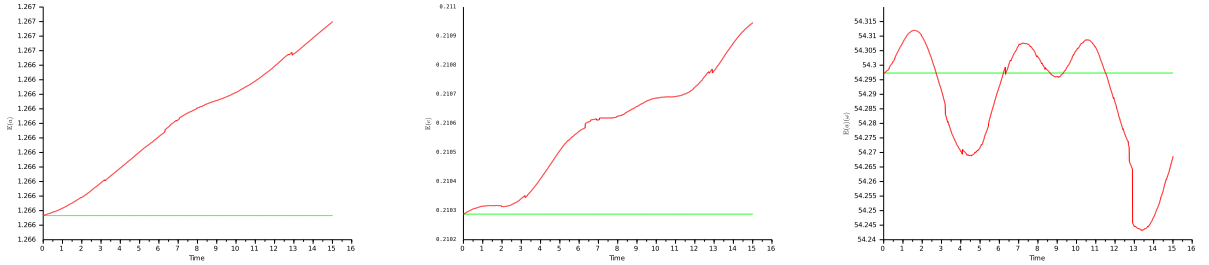
Fig. 6. Left: $\mathrm{E}(a)$. **Center:** $\mathrm{E}(e)$. **Right:** $\mathrm{E}(\omega)$.

where $\tilde{R} = \begin{pmatrix} \tilde{R}_1 \\ 0 \end{pmatrix}, \tilde{T} = \begin{pmatrix} 0 \\ \tilde{T}_2 \end{pmatrix}$ and $d\mathbf{B}_t = \begin{pmatrix} dB_t^R \\ dB_t^T \end{pmatrix}$, with B_t^R and B_t^T being independent. In our example we have $\bar{R}$ and $\bar{T}$ equal to zero, $\tilde{R}_1 = r\sigma_r$ and $\tilde{T}_2 = \sigma_\phi$.

The variation of orbital elements is derived from the well known relations (see Goldstein (2002) (3-57) p.96, (3-58) p.97 and (3-84 p.103))

$$H = \frac{-k}{2a}, \quad \frac{M^2}{mk} = a(1 - e^2), \quad \tan \omega = \frac{A_x}{A_y}, \tag{7.2}$$

where A_x and A_y are the component of the Laplace-Runge-Lenz vector in $(S, \vec{x}, \vec{y})$. Using Itô formula, we obtain the following stochastic Gauss equations in the planar case

$$da = \left[\frac{2a^{3/2}}{\sqrt{\mu(1 - e^2)}} \left(e \sin f \bar{R} + (1 + e \cos f)\bar{T} \right) + \frac{a^2}{\mu} \left(\tilde{R}^2 \left(1 + \frac{4e^2 \sin^2 f}{1 - e^2} \right) + \tilde{T}^2 \left(1 + \frac{4(1 + e \cos f)^2}{1 - e^2} \right) \right) \right] dt$$

$$+ \quad \frac{2a^{3/2}}{\sqrt{\mu(1-e^2)}} \left(e \sin f \tilde{R} + (1 + e \cos f) \tilde{T} \right) \cdot dB \tag{7.3}$$

$$de = \left[\sqrt{\frac{a(1-e^2)}{\mu}} \left(\sin f \bar{R} + \left(\cos f + \frac{e + \cos f}{1 + e \cos f} \right) \bar{T} \right) + \frac{a(1-e^2) \cos^2 f}{2\mu e} \tilde{R}^2 \right.$$

$$+ \quad \frac{a(1-e^2)}{\mu e} \left(2 - \frac{\cos f}{2} \left(\frac{2 + e \cos f}{1 + e \cos f} \right) \left(\cos f + \frac{e + \cos f}{1 + e \cos f} \right) \right) \tilde{T}^2 \right] dt$$

$$+ \quad \sqrt{\frac{a(1-e^2)}{\mu}} \left(\sin f \tilde{R} + \left(\cos f + \frac{e + \cos f}{1 + e \cos f} \right) \tilde{T} \right) \cdot dB \tag{7.4}$$

$$d\omega = \left[\sqrt{\frac{a(1-e^2)}{\mu}} \frac{1}{e} \left(-\cos f \bar{R} + \sin f \left(\frac{2 + e \cos f}{1 + e \cos f} \right) \bar{T} \right) \right.$$

$$+ \quad \frac{a(1-e^2)}{\mu e} \left[\frac{\sin 2f}{2e} \tilde{R}^2 - \frac{\sin f}{(1 + e \cos f)^2} \left(1 + \frac{\cos f}{e} (2 + e \cos f)^2 \right) \right) \tilde{T}^2 \right] dt$$

$$+ \quad \sqrt{\frac{a(1-e^2)}{\mu}} \frac{1}{e} \left(-\cos f \tilde{R} + \sin f \left(\frac{2 + e \cos f}{1 + e \cos f} \right) \tilde{T} \right) \cdot dB \tag{7.5}$$

where $\mu = \frac{k}{m}$ and the contribution due to the stochastic perturbation is written in red. We refer to F. Pierret (2013) for the general case.

8   Conclusions

The stochastic two-body problem displays a fast change of the dynamics with respect to the classical one despite the smallness of the stochastic perturbation. This result reinforces the necessity to take into account usually ignored stochastic phenomenon in order to obtain relevant predictions on the long term dynamical behavior of dynamical systems. This conclusion is fundamental for the study of the long term evolution of the solar system.

As a consequence, the following list of open problems can be studied :

- Stochastic perturbations induced by the deformation of bodies. As a first step, we would like to study a J_2-problem (see Brouwer, D. and Clemence, G. M. (1961)) with a random or stochastic J_2 constant and its influence on the rotation of the earth.

- In order to perform simulations over a very long time, we need to construct high order stochastic Runge-Kutta type integrators.

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