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PART I

0 Introduction

We attempt in the notes below to review some of the main ideas of Mathematical finance and to provide a working knowledge of its techniques via solved exercises.

0.1 Historical notes

While the words mathematical finance usually refer nowadays to the recently born field of pricing and hedging of financial derivatives, the beginnings of this science go actually far back in time, when a Japanese grain merchant invented something he called the "candles" method for predicting fluctuations in the price of grain, based on previous observations; this was the beginning of what's called today "technical analysis". Later, in the 19-th and 20-th century the forecasting needs of insurance companies have brought forth actuarial science, with its heavy reliance on statistics and probability. In the last thirty years, major upheavals were brought forth by the simultaneous emergence in 1973 of the huge market of "financial derivatives" and of a mathematical theory describing them.

This theory was ushered in by the work of P. Samuelson, who put together two very good ideas:

1. That asset prices should be modelled as multiplicative (because of compounding) Markovian processes and
2. That analytic computations work often more easily in continuous models

and came up with the favorite model of mathematical finance, **exponential Brownian motion**.

It had already been known from the beginning of century (for example from Bachelier and Einstein's work) that problems about Brownian motion reduce usually to solving associated ordinary and partial differential equations, so from that point the results started to pour. In 1968 Samuelson and Mc. Kean produced the first analytical approximation for the exercise boundary of American put options, and in 1973 Merton (Samuelson's student) and Black and Scholes came up with a very elegant solution to the problem of **rational option pricing**. This was based on the fact that companies who sell financial derivatives (whose future payoff is uncertain) create protecting "hedging" portfolios who attempt to replicate as close as possible the value they will finally have to pay to their customers. It was argued that the rational option price had to be equal to the initial value necessary to set up an "optimal hedging" portfolio.

The solution of this and other similar optimization problems have created mathematical finance as an interdisciplinary field which combines techniques of optimization, differential equations, stocastic processes and optimal control.

The first analytic solution of the optimal hedging problem for call options was obtained by Black and Scholes by solving the usual partial differential equations which emerge when solving Brownian motion problems, or, more generally, problems about continuous Markovian processes.

An alternative approach emerged later, based on an initial observation that the answer could be expressed as an expectation of the option's final payoff with respect to a certain artificial density called which came to be known as "state price density" or "risk neutral measure". Focusing on state price densities, initially motivated by the convenience of computations, turned later into an elegant approach for dealing with various type of "imperfections" in the Black Scholes model like ignoring investing constraints, transaction costs and misspecification of the model. It turned out that each type of imperfection modified somehow the state price density. This approach, called the "martingale-duality" approach, produced "robust" results stripped of any dependence on the exponential Brownian model or other models, and provides nowadays the accepted theoretical foundation for Mathematical Finance.

In the first part of these notes we consider mainly the so called "complete" case in which there is only one possible choice for the risk neutral measure. Taking expectations with respect to this measure reduces all the problems considered to problems of classical Markovian modeling, which is pretty much the same as solving various differential or integro-differential equations.

In the second part we focus on the martingale formulation of the problems and on how this allows us to deal with various types of possible "imperfections" which may arise.

0.2 On risk neutral valuation and hedging

The Black-Scholes result lead to what is nowadays known as the **risk neutral valuation** principle which states that in order to avoid "arbitrages" (market imperfections), the present value of any future "derivative" claim whose payoff $H(S_T)$ is contingent on that of a "primary" asset S_T has to be evaluated by

$$v_0 = \mathbb{E}_Q e^{-rT} H(S_T)$$

where Q must be a "risk neutralized" measure. These are measures with respect to which the expected value of the primary assew increases as if it were riskless (or as if the present value doesn't change) i.e.

$$\mathbb{E}_Q \; S_t = S_0 e^{rt},$$

where r is the rate of growth of "risk free" cash.

Note: A (very) heuristic explanation for this principle is that the hedging methods for protecting a financial derivative involve holding a position which mixes the asset S_t with risk free cash, and this somehow confers to the process S_t the expected growth rate of cash.

The risk neutral measure is typically not unique; however, in principle, it can be determined in a straightforward way once a loss function for hedging mistakes is chosen, being given by:

RN valuation theorem: The pricing (and hedging) of a financial derivative needs to be done using the RN pricing measure Q which is the closest to the observed measure of the asset P (in a sense defined by the specified loss function).

Note: Once the appropriate pricing risk neutral measure has been chosen, the value of a derivative at any intermediate moment in time may also be computed as the conditional expectation given S_t

$$v_t = \mathbb{E}_Q[e^{-rT}H(S_T)/S_t]$$

and this value in its turn determines the optimal hedging strategy. Thus, RN valuation gives not only the answer to pricing issue, but also to the hedging issue.

In conclusion, the pricing of a financial derivative may be roughly divided in three tasks:

1. Statistical Estimation: finding a good statistical model P describing the primary asset S_t.
2. Choosing an appropriate loss (utility) function and solving the corresponding optimization problem for the hedging portfolio. As stated, this leads always tothe choice of some risk neutral measure.
3. Computing expectations of various financial derivatives with respect to the chosen measure Q.

The principle of risk neutral valuation will allow us to largely ignore from now on the very important, but also very difficcult first two issues of statistical estimation and choice of an utility function. * Essentally, we assume that these tasks have been already performed, and take advantage of the fact that their answer resuts always in specifying soem risk neutral measure Q. From this point, mathematical finance becomes "pure stochastic processes": supposing that a given RN measure Q was chosen, how can we compute the expectations of the various types of claims traded in the market? The classical answer is provided by formulating and solving various differential or integro-differential equations.

With the exception of the third chapter in which risk neutral valuation is discussed in the simple context of the Cox-Ross-Rubinstein multinomial model, hedging and optimization issues will be conspicuously absent from the first part of our notes. The goal of Part I is to give the reader a working knowledge of computing expectations of functionals of Markovian processes via conditioning. In discrete time (for example for random walks) this leads to formulating difference equations. In continuous time, these become in the limit either differential equations in the case the limit is assumed to be continuous (Brownian motion), or integro-differential equations if the limit is assumed to be a general Levy process (the case when the limit is assumed to be pure jump may also be handled via renewal equations); in any case, all these types of equations are best solved by taking Laplace transforms, which converts them to algebraic equations. Hence, this part of our notes looks somewhat like a primer in differential equations and Laplace transforms.

*Of course, the statistical issue (maybe the most important) does not have a clear cut answer. In selecting the most appropriate RN measure, we are hindered both by not being able to solve the first issue and by the fact that utility is difficult to quantify.

0.3 Contents

PART I: Markovian modelling

The first chapter presents the solution to the simplest portfolio optimization problem, the one period Markowitz model.

The second chapter introduces financial derivatives and outlines their economic role.

The problem of hedging is considered in the third chapter, only within the simplest possible discrete model for asset prices evolution, the Cox-Ross-Rubinstein multinomial model. The purpose of this section is to establish the "risk neutral" representation of optimal hedging within the simplest possible context, after which we take a leap of faith and postulate that this representation holds for more complex continuous time models as well.

In the context of this simple model, it will become clear that risk neutral valuation is just a particular case of the "strong duality theorem" of linear programming.

Starting with the fourth chapter, we turn to the continuous time Markovian models of mathematical finance: Levy processes and exponential Levy processes, including the convenient Brownian motions, which have been so convenient for deriving analytical results. We also discuss here various applications. By taking the principle of RN valuation for granted, we obtain the famous Black-Scholes formula for call options. We also touch at a motivational level on the pricing of more complicated "exotic" options: "barrier", "Asian" and "American", which illustrate the need to be able to compute expectations of maxima, integrals and passage times of stochastic processes.

The fifth and sixth chapters illustrate one of the most useful features of Markov processes; the equivalence between computing expectations of various functionals and of solving associated difference/differential equations.

Starting with discrete time random walks, we show in chapter five how these difference equations are obtained by conditioning on the position of the process after one step. For continuous processes like Brownian motion, similar infinitesimal arguments presented in chapter six lead to differential equations.

Chapter seven presents some extensions and applications to the risk neutral valuation of some more complicated financial products: options on currency and on dividend yielding assets.

The last chapter of part I considers a particular type of options called **Canadian Options** which have random exponential expiration time. They have the pedagogical advantage that solving them requires solving only ordinary differential equations, as opposed to the usual options with deterministic expiration time which require solving partial differential equations. Within this special class, we are able to price analytically various types of options: barrier, American and lookback.

PART II

The first chapter in the second part is devoted to martingales, a class of processes originally studied in connection with gambling, which became very useful in finance too. We focus here on applications of the optional stopping theorem, which shows that some of the results on barrier options derived previously hold actually for a much more general class of processes.

The next chapter introduces diffusions (general continuous time Markov processes), which form the cornerstone of mathematical finance. They are defined as solutions of "Stochastic Differential equations" which are stochastic differential equations with a Brownian motion forcing term. We present here a very useful tool, Ito's lemma for general diffusions. The focus is on the special case of geometric Brownian motion.

Admittedly, the role played by martingales and by stochastic differential equations in the later sections on pricing derivatives is considerably subtler than that played in the simple applications we can cover in our preparatory sections. When differential equations and martingales finally do enter the picture, they do it so quickly that the best we are able to do then is shout: "Tighten your seat belts, martingales ahead!" We hope however that the introduction of these preparatory sections would have provided by then some psychological support for the encounter.

The third chapter turns again to the fundamental problem of portfolio optimization, this time in the context of assets modeled as exponential Brownian motion. We solve the long run growth maximization problem for portfolios of geometric Brownian motions, i.e. we derive the optimal investing strategy and the formula for the yield of a currency unit invested for optimal long run growth, for portfolios of assets assumed to follow Geometric Brownian motions.

In the chapter: More on risk neutral valuation we reexamine the general fundamental theorem of valuation of financial derivatives as discounted expectations of future values (which leads in the case of the call options to the famous Black-Scholes formula).

This chapter dwelves in more depth on issues of pricing financial derivatives in geometric Brownian motions markets, like the equivalence between the change of measure and the discounted pricing formulas (the Cameron-Martin-Girsanov change of measure). An interesting consequence is the interpretation of the risk neutral value as a discounted value with respect to the optimal performance achievable by portfolio optimization. This may be compared with the classical actuarial valuation, in which discounting is done with respect to the risk free interest. Thus, the classical actuarial discounting may be viewed as a particular case of the mathematical finance "discount by optimal portfolio performance" method, under the extra constraint that only risk free investing for the portfolio is allowed.

Finally, the last chapter **Beyond Black-Scholes: Jump-diffusion models, GARCH and Stochastic volatility models, Constraints, Transaction costs** is devoted to various attempts to remedy the defficiencies of the Black Scholes model, by considering more complex models. This will split in further chapters in due time.

1 Single period portfolio optimization

Portfolio optimization is one of the most important problems of finance. Suppose we have at our disposal I assets with prices $S_i, i = 0, 1, ...I$. The **portfolio optimization** problem is to determine proportions π_i, $\pi_i \geq 0, \sum_{i=0}^{I} \pi = 1$ in which we would split a currency unit in order to maximize our return (in some sense to be discussed later). We assume that we review the investment after some fixed period of time, at the end of which the value of assets will be given by some random variables $S_i(1 + R_i)$; R_i denote the returns per currency unit of each asset.

Suppose now that we split a currency unit in proportions π_i. Portfolio optimization is based on the following elementary equation:

The equation for the combined return R at the end of one period is:

$$R = \sum_i \pi_i R_i.$$

R is a random variable and in order to optimize its componence we will need first to obtain some estimate of the distributions of R_i over the period to be observed. At the minimum, we will need to estimate the expected returns of the assets $r_i = \mathbb{E}R_i$ and their covariance matrix $C = \sigma_{i,j} = \mathrm{Cov}\left(R_i, R_j\right)_{i,j=1,...I}$. The simplest solution to the portfolio optimization problem to be discussed below, Markowitz optimization, is based on using these estimates only.

To get some idea about what we can achieve by portfolio optimization, let us examine a plot representing the returns $R_i, i = 1, ..., I$ from several assets. We indicate only the main characteristics of each asset: the mean $r_i = \bar{R}_i$ and the standard deviation $\sigma_i = \sqrt{\mathrm{Var}\left(R_i\right)}$ (which reflects the risk associated to a stock) on a plot with axes (σ, r) (the covariances are not represented).

This plot suggests one difficculty of portfolio optimization. A rational investor would only interested in assets situated near the "North West" border of the set of available assets $(\sigma(\hat{R}_i), \hat{R}_i)$, which have both large expected return and small risk (standard deviation). However, the choice between the points near that border is not clear cut, since the assets with larger expected return have also larger risk. Depending on individual preferences, different investors will have different "optimal portfolios." Thus, what we are after is not one optimal portfolio, but rather one curve representing all the optimal portfolios for various investors preferences.

The plot above does not indicate the points $(\sigma_R, \bar{R})$ obtained by combining stocks. It is natural to expect that points $(\sigma_R, \bar{R})$ representing the standard deviation and mean of the return R of combined portfolios will lie somewhere "between" the points (σ_i, r_i) representing the single individual investments. To make this more clear, we investigate now the case when only two assets are available. We will find that when only two assets are available, by combining them in positive proportions, the investor may obtain any point lying on a curve connecting the two points which curves upwards (is concave); thus, any combination of the expected returns of the two assets may be achieved, and with a risk (standard deviation) which is smaller than that of the corresponding combination of risks.

Lemma 1.1. *Let R_1, R_2 be two given assets.*

a) The expected return and standard deviation of any combined return $R = \pi_1 R_1 + \pi_2 R_2$, $\sum \pi_i = 1$ lie on the parametric curve:

$$(\sigma_R = \sqrt{\pi_1^2 \sigma_{R_1}^2 + \pi_2^2 \sigma_{R_2}^2 + 2\pi_1 \pi_2 \rho \sigma_{R_1} \sigma_{R_2}}, \bar{R} = \pi_1 \bar{R}_1 + \pi_2 \bar{R}_2)$$

where ρ is the correlation of the two stocks.

b) When both π_i are nonnegative, the risk (standard deviation) σ_R of a combined is less than the corresponding combination of risks $\pi_1 \sigma_{R_1} + \pi_2 \sigma_{R_2}$ (obtained by connecting the two points by a straight segment).

Proof a) This follows immediately from the linearity of the expected return and from the formula for the standard deviation of a combination:

$$\sigma_R = \sqrt{\pi_1^2 \sigma_{R_1}^2 + \pi_2^2 \sigma_{R_2}^2 + 2\pi_1 \pi_2 \rho \sigma_{R_1} \sigma_{R_2}}$$

b) Using the above formula, we find that

$$\sigma_R^2 \leq (\pi_1 \sigma_{R_1} + \pi_2 \sigma_{R_2})^2$$

is equivalent to $2\pi_1 \pi_2 \rho \sigma_{R_1} \sigma_{R_2} \leq 2\pi_1 \pi_2 \sigma_{R_1} \sigma_{R_2}$ which is true since $\rho \leq 1$ and since the proportions π_i are positive.

Notes: 1) Combining investments is thus beneficial, since it reduces the risk more than it reduces the expected return.

2) While combining any two assets will reduce the risk, the reduction is greatest for negatively correlated assets. In fact, discovering assets which are negatively correlated is a highway for getting rich!

3)In the case when no nonnegativity constraints are imposed on π_i (i.e, shortsales are allowed), the resulting combined portfolios are represented by points will lie on the continuation of the curve between te two ponts described previously. Suppose for example that R_1 is the highest return asset. Taking $\pi_1 > 1$ and $\pi_2 < 0$ we can obtain points on the continuation of the curve which extends towards infinity; this means arbitrarily high expected returns accompanied by arbitrarily high risks. The selling of a low return asset in order to buy more of a high return asset is called **leverage**. Enough leverage can get the investor arbitrarily high expected returns (at the cost of arbitrarily high risks). Leverage offers another attractive method to get rich: suppose one could find two two assets with different return rates which are also almost riskless; leveraging huge amounts on the lowest return asset would then be immensely beneficial (this means borrowing at a low rate and saving at a high rate)! In practice there are of course various natural restrictions on the sign and size of the proportions which may be invested in an asset; leverage is usually impossible.

In conclusion, two important laws of investing are:

- **Combining investments (especially negatively correlated ones) is beneficial**.
- **A rational investor is only interested in combined portfolios situated on an upper curve that borders on the "North West" the set of all achievable pairs** $(\sigma(\hat{R}), \hat{R})$, **which is called the efficient frontier.**

The efficient frontier for more than two assets will be computed in the section on Markowitz optimization.

The time has come to discuss reasonable investor objectives for portfolio optimization. The first to come to mind, maximizing the expected return, is unreasonable at least for the case when shortselling is allowed, since leveraging (shortselling products with low returns and using the proceeds to buy high returns products) produces arbitrarily high expected returns (at the price of increasing the risk). This brings us to the first possible objective for portfolio optimization: **Minimization of the variance** σ_R^2 **of the combined return** R.

1.1 Minimum variance optimization:

Since

$$\mathrm{Var}\,(R) = \sigma_R^2 = \sum_{i,j} \sigma_{i,j} \pi_i \pi_j$$

where $\sigma_{i,j} = \mathrm{Cov}\,(R_i, R_j)$ we find that minimum variance portfolio optimization is a quadratic optimization problem.

$$\min \sigma_R^2 = \sum_{i,j} \sigma_{i,j}\pi_i\pi_j \tag{1}$$

$$\sum_i \pi_i = 1 \tag{2}$$

In vector notation: $\Pi = (\pi_1, \pi_2, ...)$, $C = \{\sigma_{i,j}\}_{i,j=1,..,I}$, $O = 1, 1, ...1$ we write this as:

$$\min \Pi' C \Pi \tag{3}$$

$$O'\Pi = 1 \tag{4}$$

Exercise 1:

Find the mimimum variance portfolio if $\sigma_{1,1} = 1, \sigma_{1,2} = \sigma_{2,1} = \frac{1}{3}, \sigma_{2,2} = 2$.

Solution I: Substitution We have to minimize the quadratic function: $\sigma_R^2 = \pi_1^2 + \frac{2}{3}\pi_1\pi_2 + 2\pi_2^2$ under the constraint $\pi_1 + \pi_2$=1. Using substitution ($\pi_2 = 1 - \pi_1$), the problem reduces to finding the minimum of $\frac{7}{3}\pi_1^2 - \frac{10}{3}\pi_1 + 2$. This is obtained for $\pi_1 = \frac{5}{7}, \pi_2 = \frac{2}{7}$ and yields a minimum variance of $\frac{17}{21}$.

We can also give a solution based on Lagrange's method (more convenient for many variables). Before we embark on the general case, we note:

Lemma 1.2. *The gradient of a quadratic function*

$$f(\Pi) = \sum_{i,j} \sigma_{i,j}\pi_i\pi_j = \Pi' C \Pi$$

is given by

$$\nabla f(\Pi) = 2C\ \Pi$$

The solution of the general case will be based on the following observation:

Proposition 1.3. *The solution of the quadratic optimization problem with one linear constraint:*

$$\begin{aligned} \min f(X) &= X'CX \\ b' \cdot X &= c \end{aligned}$$

is of the form $X = kZ$, where Z, k may be found in two steps:

1. *$Z = C^{(-1)}b$.*
2. *Choose a constant k so that $X = kZ$ satisfies the constraint (thus $k = \frac{c}{(Z' \cdot b)}$.*

Proof: By the method of Lagrange, we need to solve the system of the smooth fit equation and the constraint

$$\begin{aligned} \nabla f(X) &= 2C\,X = \lambda\,b \\ X' \cdot b &= c \end{aligned}$$

The solution of the first equation is

$$X = \frac{\lambda}{2} C^{-1} b$$

Note that $\frac{\lambda}{2}$ is just a proportionality constant; denoting it by k, we have $X = kZ$, where $Z = C^{-1}b$. At the next step we determine k using the constraint.

Solution II: The "simplified" method of Lagrange We need to solve the system of the smooth fit equation and the constraint

$$\begin{aligned} \nabla f(\Pi) &= 2C\,\Pi = \lambda \bar{1} \\ \Pi' \cdot \bar{1} &= 1 \end{aligned}$$

where $\bar{1}$ is a vector of ones and $C = \begin{pmatrix} 1 & \frac{1}{3} \\ \frac{1}{3} & 2 \end{pmatrix}$

1. The solution of the first equation is

$$X = kZ = kC^{-1}\bar{1} = k\frac{9}{17}\begin{pmatrix} 2 & -\frac{1}{3} \\ -\frac{1}{3} & 1 \end{pmatrix} \bar{1} = (\frac{15}{17}, \frac{6}{17})$$

where $k = \lambda/2$).

2. From the constraint, $k = \frac{1}{\sum_i z_i} = \frac{17}{21}$

Thus,

$$X = kZ = Z/(\sum_i z_i) = (\frac{15}{21}, \frac{6}{21})$$

We turn now to the more realistic case when a risk free investment S_0 with deterministic rate r (and covariance with all the other investments 0) is also included in the available investments. The minimum variance portfolio will then clearly contain only the risk free investment. This is actually a reasonable solution, which will satisfy the 0 risk tolerance of some investors. To capture also the goals of investors willing to take some risks, Markowitz proposed to minimize the risk (i.e. the variance) under the constraint of obtaining at least some specified targeted expected return $\hat{r}$. The targeted return models the risk preference of the investor.

Note: In order to represent the covariances of some stocks, the numbers $\sigma_{i,j}$ must satisfy certain inequalities, like for example the "correlation" inequality $\frac{\sigma_{i,j}}{\sqrt{\sigma_{i,i}\sigma_{j,j}}} \leq 1$, and other inequalities, which are collectively referred to by saying that the matrix C is **positive**. These conditions are precisely the positivity of all the principal determinants, which ensure that the quadratic function $\sum_{i,j} \sigma_{i,j}\pi_i\pi_j$ is convex and has thus a unique minimum. Thus, for all "plausible" covariances, the problem (4) is well posed and has a unique minimum.

1.2 Markowitz optimization

The first determination of the efficient frontier was achieved by Markowitz, who proposed to maximize the expected return, subject to an upper bound v on the variance ("risk tolerance"):

$$\max \sum_{i=0}^{I} \pi_i \bar{R}_i \tag{5}$$

$$\sum_{i,j=1}^{I} \sigma_{i,j}\pi_i\pi_j \leq v \tag{6}$$

$$\sum_{i=0}^{I} \pi_i = 1 \tag{7}$$

Note: The Markowitz problem involves a riskfree asset indexed by 0 with deterministic return $r_0 = r$.

The problem (7) turns out to be equivalent to minimizing the variance subject to a lower bound $\hat{r}$ on the expected return:

$$\min \sum_{i,j=1}^{I} \sigma_{i,j}\pi_i\pi_j \tag{8}$$

$$\sum_{i=0}^{I} \pi_i \bar{R}_i \geq \hat{r} \tag{9}$$

$$\sum_{i=0}^{I} \pi_i = 1 \tag{10}$$

Notes: 1) The proportion π_0 invested in the riskfree investment appears in both constraints, but not in the objective of (10). This will facilitate later removing it altogether from the problem.

2) The Markowitz formulations capture the idea that portfolio optimization involves a **tradeoff between expected returns and risk**.

Using the method of Lagrange, we see that the two approaches above are equivalent, and furthemore they are equivalent to maximizing:

$$\max \sum_{i=0}^{I} \pi_i \bar{R}_i - \lambda \sum_{i,j=1}^{I} \sigma_{i,j} \pi_i \pi_j \tag{11}$$

$$\sum_{i=0}^{I} \pi_i = 1 \tag{12}$$

for some fixed λ; this parameter expresses the "risk-return trade-off" of the investor.

Notes: 1) The objective $\mathbb{E}R - \lambda Var(R)$ in this formulation maybe interpreted as a "risk penalized" expected return.

2) The Lagrangian formulation has the advantage of putting in evidence the fact that the roles of the objective and the constraint are symmetric. The disadvantage is that it requires inputting the "risk-return trade-off" parameter λ which is more difficcult to interpret than either the targeted return $\bar{r}$ or the "risk tolerance" v.

When either λ, $\bar{r}$ or v vary, the solutions will trace the same curve, called efficient frontier.

To stress the analogy with the previous section, we will work with the formulation (10). The problem (10) can be solved by studying the Lagrangian equations $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$, where g_1, g_2 are the two constraints.

It is possible however to simplify the problem first and get rid of the constraints altogether, in two steps:

1. As noted, the optimization objective does not actually depend on π_0. We proceed now to eliminate π_0 from the first constraint, by substracting r times the second constraint from the first one. Dropping the second constraint (on the sum of the proportions being 1) altogether we arrive at the following **reduced problem** which involves only the proportions of the risky assets:

$$\min \sigma_r^2 = \sum_{i,j=1}^{I} \sigma_{i,j} \pi_i \pi_j \tag{13}$$

$$\sum_{i=1}^{I} \pi_i (\bar{R}_i - r) \geq \hat{r} - r \tag{14}$$

2. It is clear (from our experience!) that the minimum risk will happen when the constraint is satisfied with equality. We are then in precisely the situation of Proposition 3.3, with the vector of coefficients of the constraint being $\tilde{R} = (\bar{R}_1 - r, \bar{R}_2 - r, ...)$.

   Namely, the method of Lagrange leads to the following system for $\Pi = (\pi_1, \pi_2, ...)$:

$$\sum_{j=1}^{I} 2\sigma_{i,j} \pi_j = \lambda (\bar{R}_i - r), i = 1, .., I$$

or in vector form

$$C\Pi = \lambda \tilde{R}/2,$$

where $\tilde{R} = (\bar{R}_1 - r, \bar{R}_2 - r, ...)$ is the vector of excess returns over the risk free interest. As in Proposition 3.3 above, the solution must be of the form $\Pi = kZ$, where

(a) We find Z from the matrix equation $CZ = \tilde{R}$.

(b) To satisfy the constraint $\Pi'\tilde{R} = \hat{r} - r$, we must have $k = \frac{\hat{r}-r}{Z'\tilde{R}}$.

In conclusion, the problem of finding the optimal investment proportions in the presence of interest rates has been decomposed in three steps:

1. Solve the equations
$$CZ = \tilde{R}.$$
2. Let $\Pi = kZ$, where $k = \frac{\hat{r}-r}{Z'\tilde{R}}$.
These are the optimum proportions to be invested in the risky assets.
3. Find $\pi_0 = 1 - \sum_{i=1}^{I} \pi_i$, to be invested in the riskless asset.

In practice, we usually determine only a portfolio Z^* comprised only of risky assets (thus, at step 2, we normalize by the sum of the components of Z), called **pure risky efficient portfolio**.

The reason is that

Lemma 1.4. *In the presence of a riskless investment, the efficient frontier is a half line obtained by combining the riskless investment with the pure risky efficient portfolio, in some proportions which depend on the investor's expected return target.*

Since those proportions are best left to be decided by the investor, it is enough if the financial engineer determines the pure risky efficient portfolio.

Note: The policies described in this section of keeping some constant proportion π in the stock over a multi period horizon are referred to as **dynamic rebalancing**. Note that in order to keep constant proportions, intensive trading will be in general required; the stocks which go up will have to be trimmed down, and the holdings which went down will have to be increased (which is in keeping with the traditional sell high/ buy low). While these policies achieve much better long run returns than say deciding initially on some fixed proportions and then never rebalancing, they also involve substantial trading and thus large transaction costs may be incurred.

We end this section by displaying graphically some examples of efficient frontiers $(\sigma(\hat{r}), \hat{r})$, when $\hat{r})$ ranges over all possible targeted returns $\bar{r} \geq r$, and $\sigma(\hat{r})$ denotes the minimum standard deviation achievable for a given targeted expected return $\hat{r}$.

Example 1: Suppose $r_1 = .2, r_2 = .4, r = .1$ and the covariance matrix is given by:
$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

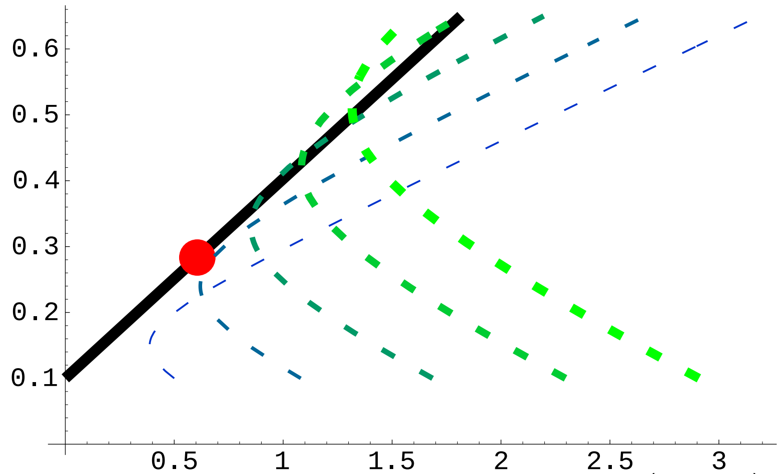

Figure 1: **positively correlated stocks** ($\rho = .5$)
$W = (-.06, .33, .73)$; the small dots are the avilable stocks and the big dot represents the portfolio recommended by the continuous approximation

Example 2: Suppose $r_1 = .3, r_2 = .6, r = .1$ and the covariance matrix is given by: $\begin{pmatrix} 1 & -.8 & 0 \\ -.8 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

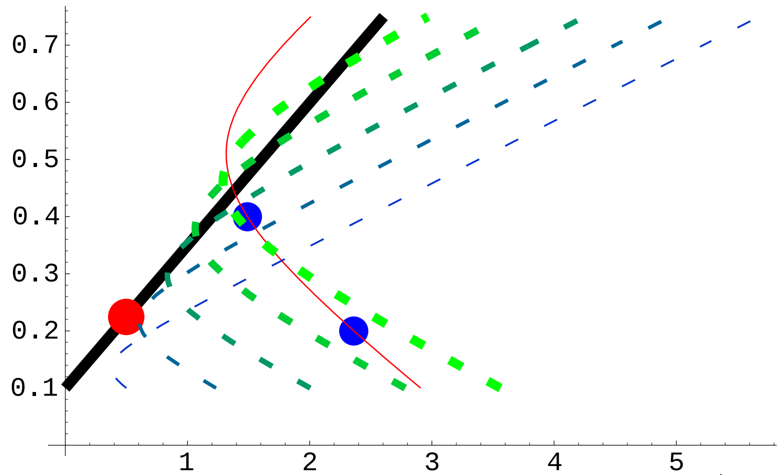

Figure 2: $\rho = -.8$ **negatively correlated stocks!** $W = (3.33, 3.66, -6)$

Example 3: Suppose $r_1 = .6, r_2 = 1, r = .1$ and the covariance matrix is given by: $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

1.3 Markowitz optimization with shortsales constraints

1.4 "Robust" optimization **

As evidenced by the Markowitz approach, the optimal portfolio depends on investor preferences. These are often expressed via maximizing expected **utility functions**. Two popular

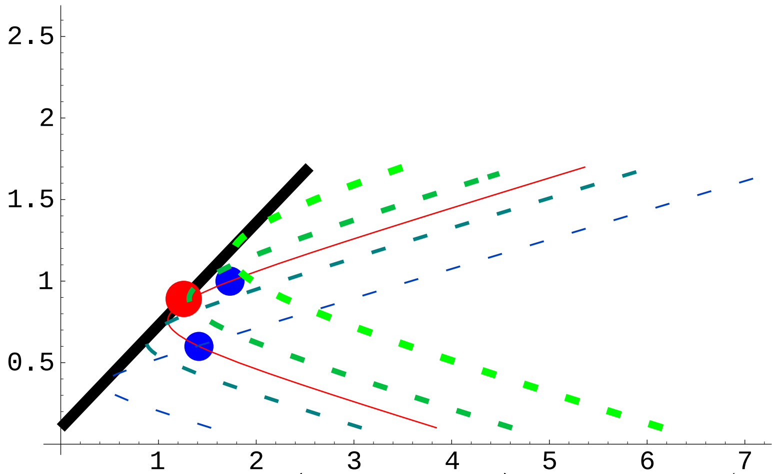

Figure 3: **independent (uncorrelated) stocks.** $W = (.5, .6, -.1)$

such classes of functions are: $\frac{\mathbb{E}(1+R)^\theta - 1}{\theta}$, which becomes in the case $\theta = 0$ $\mathbb{E}Log(1+R)$ and $\mathbb{E}e^{\theta R}$. When θ varies, the solution of either problem will trace the same efficient frontier obtained by the Markowitz approach. However, the "correct" θ expressing a given investor's preference is purely an abstract concept, which can not be determined reliably in practice. An interesting problem raised by Sornette is to determine a point on the efficient frontier which is "robust" to changes in the investor's attitudes (for example, which is within a small distance of the optimum, for a whole range of investor's utilities). More precisely, Sornette proposes to determine a portfolio for which both the variance and the fourth order cumulant of the returns are small.

1.5 Exercises

Exercise 1.1

Find the minimum variance portfolio in the case of I **independent** risky assets.

Exercise 1.2

Find in the case of I **independent** risky assets the portfolio that minimizes the **kurtosis**

$$\frac{\mathbb{E}(R-\bar{R})^4}{(\mathbb{E}(R-\bar{R})^2)^2}.$$

Exercise ** 1.3

Find in the case of I **independent** risky assets the portfolio that minimizes the normalized cumulant of order six.

Exercise ** 1.4 Find the formula for the efficient frontier in the first example above

a) if no riskless investment was possible b) with the riskless investment included c) plot both curves.

Exercise ** 1.5 Find the formula for the efficient frontier for two perfectly correlated assets, if no riskless investment is possible.

Exercise ** 1.6 Find the formula for the efficient frontier for two perfectly uncorrelated assets, if no riskless investment is possible

Exercise 1.7 Find the optimal investment policy for an opportunity set including a riskless investment with rate $r = 6\%$ amd three risky assets with respective expected returns $14\%, 8\%, 20\%$, standard deviations $6\%, 3\%, 15\%$ and correlations $\rho_{1,2} = .5, \rho_{1,3} = .2, \rho_{2,3} = .4$, if no shortsales are allowed.

1.6 Solutions

Solution 1.1

By Lagrange's method, the solution of

$$\min \sum \pi_i^2 \sigma_{i,i}$$
$$\sum \pi_i = 1$$

is when $\pi_i \sigma_{i,i}$ is constant; hence, $\pi_i = \frac{\sigma_{i,i}^{-1}}{\sum_j \sigma_{j,j}^{-1}}$.

2 Background on financial derivatives

In this chapter we describe some financial derivatives (also called options or claims), review their history and discuss their uses.

Options are generally defined as contracts between two parties in which one party has the right but not the obligation to do something at a final later time T, usually to buy or sell some underlying asset S_T under protected conditions. Having rights without obligations has financial value, so option holders must purchase these rights, making them assets. This assets derive their value from the primary asset S_T, so they are called derivative assets. More generally, financial derivatives may be viewed as random future payoffs H_T which depend somehow on the price of the primary asset, i.e. $H_T = f(S_T)$. Payment for these options takes the form of a flat, up-front sum called **premium**.

For example, one of the most used derivatives is the **call** option, which gives to the option holder (buyer) the right to buy an asset with price S_t at a later "expiration" time T and at a predecided "reserved" **exercise price** K. Thus, the effective final payoff to the option holder is

$$H_T = (S_T - K)_+ = \begin{cases} S_T - K & \text{if } S_T \geq K \\ 0 & \text{if } S_T \leq K \end{cases}$$

(since the option will not be exercised if the asset's price drops below K).

An investor would buy a call option if he forecasts that the price at T of the asset S_T will be larger than K. This could be for example a way to ensure that he can buy later the asset (at price K), despite its increase in price. Of course, instead of buying the call option, the investor could buy the asset in advance, but this would commit him to holding this asset; by buying the call option (whose price is typically just a small fraction of the asset's price), he can later give up holding the asset (if its price drops below K.) The issue is what should be the present value of (or initial price) for such a contract. As a starting estimate, we could assume that the price won't change, which gives the value $(S_0 - K)_+$ as a rough approximation (since the exercise price K is only payed at expiration, this leaves only $S_0 - K$ to be payed upfront, and this only in the case that $S_0 \geq K$).

For a more sophisticated answer, we will need to incorporate somehow in our anaswer both our view of the final value of S_t, and of "to what extent the uncertainty in whether the option will be exercised or not" will be hedgeable.

Some of the most traded options are:

- **Call options** with payoff $(S_T - K)_+$
- **Put options** with payoff $(K - S_T)_+$
- **Straddle options** with payoff $(K - S_T)_+ + (S_T - K)_+$
- **Binary, or digital options** with payoff $1_{\{S_T \geq K\}}$
- **Spread options** with payoff $1_{\{K_1 \leq S_T \leq K_2\}}$.

Note that the buyer of a call option or binary option is betting on the future price of the stock ending above K, the buyer of a put option is betting on the price ending below K, the straddle is a bet on movements away from K and the spread option is a bet on a precise interval for the final prize. In principle, any arbitrary function $H_T = f(S_T)$ can be used as basis for a traded option (and some are!). The financial interpretations of derivatives is not important in the development of a pricing formula; this will be obtained in the greatest generality for an arbitrary **final payoff** function $H_T = f(S_T)$.

However, the success of a certain option on the market will depend of course on the role it plays. The put for example is very important for stock insurance. Note that the holder of a stock share and a put will end up with the payoff

$$S_T + (K - S_T)_+ = \min(S_T, K)$$

and will thus be protected from collapses in price.

2.1 The use of financial derivatives

The idea of options is certainly not new. Ancient Romans, Phoenicians and Greeks traded options against outgoing cargoes from their local seaports.

In today's world, the need to trade financial derivatives arises when individuals or companies wish to buy an asset or commodity in advance. For instance, an airline may wish to buy fuel in the future for a fixed price determined now, in order to avoid being subject to price fluctuations. This is also a big factor in foreign exchange. If you trade with another country you are subject to exchange rate fluctuations. By buying forward you can insure that you can sell the product for a certain price. Thus the idea of a *forward* was introduced, an agreement reached between two parties for the delivery of some commodity or stock in the future. There are two parties to any derivatives contract, the seller and the buyer: the buyer of the asset is said to take a long position and the seller is said to take a short position.

The trading of options is a recent phenomenon. In 1973 the Chicago Board of Trade Options Exchange was opened for the trading of options on stocks. Prior to this individuals who wished to purchase stock options would have to do so over-the-counter (OTC) from a bank. The CBOT was the first such trading organization and there are now many places which conduct trade in stock options. In London the option trading exchange is called LIFFE, London International Financial Futures Exchange. You can find LIFFE option prices quoted in the Financial Times.

There are three types of traders who deal with these products. Firstly the hedgers, people who buy options as a form of insurance against adverse market movements. If you are a company which trades with another country then you may take out an option on currency which would pay you a certain amount if the exchange rate went heavily against you. In this way you can hedge or protect your position to some extent by buying an option. For instance if you import 1 billion yen worth of Japanese electronic goods and the current pound/yen rate is 181.23, you could take out an option that will pay 1 billion yen if the exchange rate is 200 in a years time. This means that if the cost of the yen gets too high then you can cash the option and buy your goods. If the exchange rate stays low then you lose the money spent on the option and use the exchange rate which is at a rate you are prepared to pay. The cost of the strategy is that of buying the options and they will be reasonably cheap as a large currency fluctuation will be unlikely.

The second type are the speculators who essentially gamble on the way various assets will move. They take a position in the market based on their beliefs and make or lose money based on (essentially) chance. They are interested in financial derivatives as there is substantial gearing. This means that it is possible to make or lose a lot more money by buying or selling these products rather than the underlying asset. For instance a stock costs 10 pounds to buy and so you could invest 10000 pounds by buying 1000 shares. Then if the price was 11 pounds in a years time you would have made 1000 pounds. However if you had bought share options with a strike price of 10 pounds, then these would cost say 0.5 pounds, so you could buy 20000. If the price went up to 11 pounds you could exercise your option, buying 20000 shares at 10 pounds and then selling them for 11 pounds each to make a net profit of 10000 pounds. Of course if the price had dropped to 9.99 you would lose all your money!

The final group of traders are the arbitraguers. People who watch the market and try to find situations where there are risk free profits to be made. These are realized by synthesizing a product in one market with products in another so that any price discrepancy guarantees a profit. A simple example is where the price of a stock traded on two exchanges differs. If a stock is trading for 100 pounds in London, for 285 D-marks in Frankfurt and the exchange rate is 2.82 DM/£, then the price is too cheap in London. We could buy 1000 shares in London and sell 1000 in Frankfurt and convert the D-marks into pounds to make 1063.83 pounds without any risk. These opportunities are rare and as soon as they appear are driven out of existence by people seizing the opportunity to make money. As people buy stock the price will increase in London and as they sell the price will decrease in Frankfurt until the stock has the same value in the two places.

2.2 The coming of age of mathematical finance

In 1973 there occurred also a key event in the development of financial mathematics, when Myron Scholes and Fischer Black published a paper which showed how to price and "hedge" (i.e. manage a portfolio which enables the option issuer to fulfil his obligation) the European call option. In 1997 Scholes and Merton, who also contributed to the initial formulation of derivative pricing theory, were awarded the Nobel prize in economics. Their theoretical work has had a profound impact on the way the world's financial markets operate.

Historical note: "It was an ordinary autumn afternoon in Belmont, Mass. 1969, when Fischer Black, a 31 year old independent finance contractor, and Myron Scholes a 28 year old assistant professor of finance, at MIT hit upon an idea that would change financial history. Black had been working for Arthur D. Little in Cambridge, Mass., when he met a colleague who had devised a model for pricing securities and other assets. With his Harvard Ph.D. in applied mathematics just five years old, Black's interest was sparked. His colleague's model focused on stocks, so Black turned his attention to options, which were not widely traded at the time. By 1973, the tandem team of Fischer Black and Myron Scholes had written the first draft of a paper that outlined an analytic model that would determine the fair market value for European type call options on non-payout assets. They submitted their work to the Journal of Political Economy for publication, who promptly responded by rejecting their paper. Convinced that their ideas had merit, they sent a copy to the Review of Economics and Statistics, where it elicited the same response. After making some revisions based on extensive comments from Merton Miller (Nobel Laureate from the University of Chicago) and Eugene Fama, of the University of Chicago, they resubmitted their paper to the Journal of Political Economy, who finally accepted it. From the moment of its publication in 1973, the Black and Scholes Option Pricing Model has earned a position among the most widely accepted of all financial models."

Black and Scholes and Merton had succeeded to price options only under an idealized model called geometric Brownian motion market, previously proposed by the MIT economist Samuelson. They left unanswered several important issues arising in real markets, like:

- Imperfect information (unknown mean and volatility)
- Discrete trading
- Transaction costs

In later mathematical developments, the original theory was greatly extended to answer to these issues, the key turning out to be an approach called "martingale duality" or "risk neutral pricing". The first hints at this approach came with the appearance of the Cox-Ross-Rubinstein multinomial model, which will be discussed in section 2.

2.3 The replication of derivative contracts

The fundamental question about derivatives is what should be their premium, i.e. the value today for a contact which will pay some function $f(S_T)$ at a later time. **How much should people pay now for future prospects?**

Example 1: Forwards Consider a forward, which is a contract to deliver a stock at some time T in the future. One possible candidate for premium would be $v_0 = \mathbb{E}S_T$, where $\mathbb{E}$ is expectation with respect to some estimated statistical model. By the law of large numbers, this would work alright in the long run for the seller, provided the estimated model is correct. Sometimes the seller would win and sometimes they would lose, and this would be kind of a "financial roulette" for high level bank executives.

However, this entertaining roulette is played in practice only by the buyers, since a much more sensible strategy exists for the sellers. By charging a premium S_0, they can buy the stock now at time 0 and keep it ready for delivery until the end and thus fulfil their obligation at time T whatever the price then. By creating what is called a **hedging portfolio** or **"replicating" portfolio** they have eliminated any risk on their part! Clearly, if a hedging portfolio exists, then the right price for an option should be the initial expense necessary to set up the replicating portfolio, disregarding any possible statistical expectations $\mathbb{E}S_T$ we might have of the future. (Another argument in the favor of abandoning conjectured expectations is that if someone has strong feelings or insider info about the way S_t will evolve, he might as well buy the stock itself.)

Exercise What should be the premium for a forward, if the payment is done at time T, but decided already at time 0?

Solution: The price should still be $S_0 e^{r\,t}$, where r is the interest rate (assumed to be constant). The reason is that the seller's hedging strategy is still to buy the stock at time 0 at the price S_0 and hold it until the end, when its value would become $S_0 e^{r\,T}$.

Until 1973 it was considered impossible however to replicate call options and other contracts, so it looked still plausible that pricing should be done by estimating some statistical model for S_t, and the premium should be $\mathbb{E} f(S_T)$ for interest rate $r = 0$, or, more generally, the present value $e^{-r\,T}\ \mathbb{E} f(S_T)$ of the expected future payoff.

However, as shown in 1973 by Black and Scholes, under certain conditions defining an idealized type of market called **complete**, the European call option could be "replicated" (or "hedged"), by using a portfolio combining the stock and a riskless cash investment. Later, this was shown to be true for any European option by Merton. This meant that a certain initial premium v_0 could be invested and then dynamically managed throughout time such that the resulting "replicating" portfolio will end up with the final value at T which equals **exactly** $f(S_T)$ under **any** evolution of the prices, with no risk involved!

Definition A **replicating portfolio** for an option on an asset S_T with final payment $f(S_T)$ is a combination of a number of stock units Δ_t and a loan L_t whose total value $V_t = \Delta_t S_t + L_t$ will equal the value of the final claim under any evolution of the market, i.e. $V_T = f(S_T)$.

Notes: 1) The replication of a forward involves just acquiring it initially and holding it continuously until T, i.e. $\Delta_t = 1$ for any t.

2) Replicating of call options requires figuring out whether the option will end up "in the money" (in which case we need $\Delta_T = 1$) or "out of the money" (in which case we need $\Delta_T = 0$). Black and Scholes had found a hedging recipe which always kept some fraction $0 \leq \Delta_T \leq 1$ in the stock (which reflected the current chances of ending in the money), which could be "nudged" to end up exactly at one if $S_T > K$ and at 0 otherwise.

Black & Scholes and Merton were the first to show that call options may be priced by solving the following optimization problem: construct a judiciously managed "hedging" portfolio W_t which contains "optimally" chosen proportions of the risky asset S_t and of a "riskless" cash investment with fixed interest r, in such a way that the hedging portfolio W_T ends up as close as possible to the claim at the expiration time T(i.e. $W_T \approx H_T$.)

In fact, Black & Scholes showed that if the evolution of the asset S_t could be described by a stochastic process called geometric Brownian motion (of known volatility), and various complications like transaction costs and constraints were ignored, then it was possible to construct a hedging portfolio which would replicate **exactly** the value of the call option, "without any risk" (i.e. $W_T \equiv H_T$.)

The initial value W_0 of the hedging portfolio provided thus in the "Brownian" world a "no risk" initial value to be charged for a future random payment! The importance of this "miraculous" **exact "replication"** was obvious from the start, both in academic and "practitioner's" circles, who embraced the Black Scholes formula.

While the mathematics was there, it's meaning became apparent only with the introduction of the Cox-Ross-Rubinstein multinomial model (1976), which is described in more detail in Apendix A.

These economists considered a discrete time evolution of asset prices in which at each possible stage the future price was restricted to take values only out of a finite set of possibilities ("scenarios"). This brought forth the realization that these models, to be called **incomplete**, did not in general allow for exact replication, except in the **binomial case** when the number of future possible scenarios was restricted to 2. In this case, to be called **complete**, exact hedging is possible **for any type of claims** H_T. Completeness was thus a result of severely restricting the stochastic model for the future (the Brownian model may also be thought in a limiting sense, based on its "derivative" to restrict essentially the possible future scenarios to only two, one in which its "derivative" is ∞ and one in which it is $-\infty$).

Financial mathematics research focused in the beginning on the complete models. While ignoring any type of "frictions" (incomplete information, transaction costs, etc), these models were able to yield exact hedging and pricing solutions for a wide variety of financial products. Furthemore, it turned out that the same formula, called **risk neutral valuation**, could be used for the pricing of any derivative claim.

RN valuation in complete markets states that the initial value for any final claim H_T should be:

$$\mathbb{E}_Q \; e^{-rT} H_T$$

where r is the risk free interest rate of the market and Q is a measure close in some sense to the original measure (absolutely continuous with respect to it) but having in addition the property that under this measure the asset values have expectations which increase as if they were riskless, i.e.

$$\mathbb{E}_Q \; S_t = S_0 e^{rt}$$

Furthemore, the value at any time of the optimally managed hedging portfolio should equal the conditional expected value of the final claim with respect to the measure Q. (Thus knowledge of the measure Q answers both the pricing and the hedging problem).

In the nineties, the research turned towards the more realistic incomplete models in which exact replication is impossible and there always has to be a final "mishedge" $W_T - H_T$. The seller and buyer of an option naturally disagree in their preferences on the distribution of this mishedge and pricing is possible only after they manage to choose a joint common goal of minimizing some "penalty" of the mishedge $U(W_T - H_T)$. Thus, hedging and pricing in incomplete markets amounts to solving a collection of portfolio optimization problems with arbitrary final target H_T and arbitrary objective $U(x)$. **MEMP: Minimize the expecte**

$$\min_{x,\pi} \mathbb{E}_{\{W_0 = x\}} \; U(W_T - H_T)$$

with respect to the initial investment x and the proportion π which is to be invested in the risky asset S_t.

The solution of this problem, known as **risk neutral valuation in incomplete markets**, states that for a large class of penalty functions $U(x)$, the solution of the optimization problem MEMP is given by:

$$x = \mathbb{E}_Q \; e^{-rT} H_T \quad \textbf{Risk neutral valuation}$$

where Q is the measure which is closest with respect to some "dual" distance (which depends only on the penalty U) to the estimated measure P of the underlying asset. An example illustrating this duality is given at the end of Appendix A.

2.4 Examples of financial derivatives

By the RN valuation principle, every derivative product should be valuated as an expectation of the final payoff with respect to some (RN) measure. We will give now a list of the types of expectations needed to evaluate some commonly traded derivatives.

According to whether the payoff depends on the whole path of the price or on the final payoff only, derivatives may be divided in path dependent or European. The particular type of path dependent options in which the buyer is allowed to choose also the moment of termination of the contract is called American options.

Examples of European options

- **Digital options** with payoff $1_{\{S_T \geq K\}}$
- **Asset or nothing options** with payoff $S_T 1_{\{S_T \geq K\}}$ **Call options** with payoff $(S_T - K)_+$
- **Put options** with payoff $(K - S_T)_+$
- **Butterfly options** with payoff $N\ 1_{\{K-\frac{1}{2N} \leq S_T \leq K+\frac{1}{2N}\}}$.

where S_T is the value of the stock at the "expiration" time of the contract T and K is the "exercise" price. Note that the buyer of a call option or binary option is betting on the future price of the stock ending above K, the buyer of a put option is betting on the price ending below K, and the buyer of a butterfly option is betting on a precise interval for the final prize.

Analytical valuation of European options requires the availability of formulas for the Q distributions of the stock process at a fixed time.

Examples of Barrier options

- **Perpetual down and out digital** with payoff $1_{\{L \leq S_t, \forall t \in 0,T\}}$
- **Perpetual double barrier digital** with payoff $1_{\{L \leq S_t \leq U, \forall t \in 0,T\}}$
- **Down and out Call** with payoff $(S_T - K)_+ 1_{\{L \leq S_t, \forall t \in 0,T\}}$
- **Double barrier call** with payoff $(S_T - K)_+ 1_{\{L \leq S_t \leq U, \forall t \in 0,T\}}$

where L, U are fixed barriers.

Analytical valuation of barrier options requires the availability of formulas for the Q probability of reaching a barrier (for perpetual options) and of the Q distribution at a fixed time of the "absorbed" stock process (for fixed period options).

Examples of American options

- **Perpetual American put** with payoff $(K - S_\tau)_+$ **Down and out American call** with payoff $(S_\tau - K)_+ 1_{\{L \leq S_t, \forall t \in 0, \tau\}}$

where τ denotes a stopping time.

The analytical valuation of American options requires the availability of formulas for the distribution of hitting times and also that of the joint distribution of the hitting times and the hitting position.

3 Risk neutral valuation in the Cox-Ross-Rubinstein model

Paradoxically, the Black Scholes solution of the hedging problem was first provided under a quite complex mathematical model for asset prices evolution, the exponential Brownian motion model. This solution contained an enticing, though clearly unrealistic feature: the possibility under the exponential Brownian motion model to hedge options **exactly**, with no risk to the seller.

Puzzled by this feature, several prominent economists discussed at a conference in 1976 the "mystery" behind this exact hedging, and came up with a much simpler approach and pricing formula.

They considered a discrete model with finitely many scenarios allowed at each stage, known nowadays as the Cox-Ross-Rubinstein model. The conclusion was that perfect hedging was possible only if the number of future scenarios allowed at each stage was restricted to two (the "binomial" model), and ceased to be true for more then two scenarios. In the latter more realistic case, several different solutions of the problem were possible, depending on the objective chosen for hedging; a "seller" hedging, a "buyer" hedging, a least squres hedging, etc. could be defined.

Thus, the only multinomial markets in which perfect hedging is possible are binomial; this type of markets are called **complete** and for some reason to be discussed later, the Brownian motion model is complete just like the binomial model, even though the number of future possible states it allows after any time interval is infinite!

In this section we present the hedging of options under the discrete Cox-Ross-Rubinstein model. We will consider four different optimal hedging problems: the binomial (two scenarios) problem, the seller problem, the buyer problem and a least squares hedging problem, and show that in all four cases the initial value of the hedging portfolio may be computed via a recipe to be called **risk neutral valuation**). This states that the value of the optimal hedging portfolio corresponding to the various types of possible objectives can always be expressed as an expectation with respect to a certain type of measures called **risk neutral**.

In this simple context it will be clear that risk neutral valuation is just a particular case of the "strong duality theorem" of linear programming.

3.1 Hedging in discrete models

Let us denote by s_0 the initial price of a stock and by S its value after one time period.

In the Cox-Ross-Rubinstein model it is assumed that S can only take values out of a finite set of possible values: for example, think of three possible "most likely" scenarios one in which the stock moves to a higher value s_u, one in which it moves to a lower value s_d and one in which it moves to a middle value s_m.

Our market also contains a **financial derivative** (option). This is a contract ("claim") which upon expiration ensures that its holder receives a payment H whose value depends (is contingent) on that of the stock: the payoff at the end of the period is either h_u, h_d or h_m depending on whether the stock price went up, down or to the middle value. We would like to find a reasonable initial price which the buyer of this financial derivative should pay to its seller at the beginning of the period.

Of course, for practical applications it is very important to decide how many possible scenarios to use and what future values to predict for the stock value. We will ignore these practical issues however; we will assume that fixed values s_u, s_d, s_m are given (maybe enforced by law!) and we will focus on the mathematical consequences of this for pricing the option.

Definition: A hedging portfolio is a combination of a number φ of stock units and a cash investment (or loan) ψ (to be acquired by the seller) whose total combined value at the expiration time T is designed to be "as close" as possible to the value of the claim.

The initial value of the hedging portfolio, which is:

$$v_0 = \varphi s_0 + \psi$$

is then a quite reasonable price to be charged to the buyer. Usually the cash investment ψ is negative, and is thus a loan; it allows the seller to buy a larger number of stock units than could have been bought without using it.

To emphasize ideas, we assume at first the interest rate to be $r = 0$. In this case, the value of the loan remains unchanged and the value of the hedging portfolio at the end of the period will be

$$V = \varphi S + \psi$$

(where S is the random value of the stock). We'll call this the **value evolution** equation.

Sometimes, it is convenient to eliminate the loan ψ from this expression by using the equation: $\psi = v_0 - \varphi s_0$. Plugging this in (4.3.2) leads to the equivalent form:

$$V = v_0 + \varphi(S - s_0)$$

also called the **capital gains equation** since the value of the portfolio after one period is expressed as the sum of the initial value and the "capital gains" term $\varphi(S - s_0)$. Thus, our purpose is to choose the hedging portfolio (φ, ψ) so that V will be close as possible to H in some sense (yet to be defined).

$$V = v_0 + \varphi(S - s_0) \approx H \tag{15}$$

Sometimes we write instead of (15)

$$v_0 + \varphi(s_w - s_0) \approx h_w$$

where w stands for either of the possible scenarios ("up", "down", etc).

Note that the exact equality of H and V would require satisfying k equations, where k is the number of possible scenarios for the stock's evolution, and that we only have at our disposal two unknowns (φ, ψ) (or (φ, v_0). We could try to satisfy the k equations in a least squares sense, but this is by no means the only choice. For this reason, we will consider first a "toy" model in which $k = 2$ which allows one to determine the portfolio (φ, ψ) in a clearcut manner.

3.2 The one period binomial model

In this section we assume that at the end of the period the stock

may only move to one out of two values s_u, s_d.

Under this assumption it is possible to satisfy the hedging equations exactly, *whatever happens to the stock price!* Indeed, at the end of the period, the value of the hedging portfolio and the claim are respectively

Hedge	Claim
$v_u = v_0 + \varphi(s_u - s_0)$	h_u
$v_d = v_0 + \varphi(s_d - s_0)$	h_d

We need to solve thus a system with two equations and two unknowns:

$$v_0 + \varphi(s_u - s_0) = h_u \tag{16}$$

$$v_0 + \varphi(s_d - s_0) = h_d \tag{17}$$

The system (17) is of course quite easy to solve. We will emphasize however the method of **reduction** which eliminates the variable φ from the left hand side; for this we employ two row multipliers for the equations (also called in linear programming dual variables) q_u, q_d, chosen so that the coefficient of φ vanishes. Thus, q_u, q_d must satisfy

$$q_u(s_u - s_0) + q_d(s_d - s_0) = 0 \tag{18}$$

We also assume for conveniency that the multipliers satisfy

$$q_u + q_d = 1,$$

which also allows us to view them as probabilities (at least if they are positive). The implication of these two restrictions on the row multipliers is that **when combining the equations we get a formula for the initial value** v_0 :

$$v_0 = q_u h_u + q_d h_d \tag{19}$$

This equation has the nice interpretation that the initial value which makes hedging exact is an average of the possible values of the final claim H with respect to an "artificial" set of probabilities $Q = (q_u, q_d)$ (yet to be determined).

$$v_0 = \mathbb{E}_Q H$$

We call Q a set of "artificial" probabilities, since it does not reflect any observed frequencies; basically, it represents a way of expressing the result of our optimal hedging problem.

Moreover, the equation for the artificial probabilities (18) may be rewritten as

$$q_u s_u + q_d s_d = s_0 \tag{20}$$

which has also an interesting interpretation: the "Q" expectation of the stock price after one period equals precisely its initial value.

$$\mathbb{E}_Q S_1 = s_0 \tag{21}$$

Definition: A measure (i.e. set of probabilities) Q satisfying the equation (21) is called a **risk neutral measure** or "balancing" measure for the stock price.

One point left uncleared is whether the numbers (q_u, q_d) are positive.

Solving the system

$$\begin{aligned} q_u s_u + q_d s_d &= s_0 \\ q_u + q_d &= 1 \end{aligned}$$

we find that

$$\begin{aligned} q_u &= \frac{s_0 - s_d}{s_u - s_d} \\ q_d &= \frac{s_u - s_0}{s_u - s_d} \end{aligned}$$

and so both (q_u, q_d) are positive iff $s_d < s_0 < s_u$. However, models not satisfying this condition are not interesting, because they allow **arbitrage** which means the possibility of infinite profits: indeed, if both $s_0 < s_d < s_u$ a hedging portfolio with $\varphi = \infty$ would reap infinite profits and the same would be true by shortselling $\varphi = -\infty$ in the case $s_d < s_u < s_0$.

In conclusion, we obtained for the binomial model the

Theorem 3.1. Risk neutral valuation theorem: *Under the assumption of noarbitrage $s_d < s_0 < s_u$, the initial value which makes perfect hedging possible may be expressed as an expectation $\mathbb{E}_Q H$ of the final claim with respect to the (unique) risk neutral measure Q.*

Note: In this simple case, the original hedging problem of finding φ, ψ may anyway be solved directly quite easily, yielding

$$\varphi = \frac{h_u - h_d}{s_u - s_d}, \psi = \frac{h_d s_u - h_u s_d}{s_u - s_d}$$

However, in more complicated situations, risk neutral valuation (i.e. the determination first of the measure Q comprised of the row multipliers) becomes by far the easiest method for determining the initial value and the hedging strategy.

Exercise: Develop the **CRR model with non zero interest rate r, over a period of length t.**

Solution In the general binomial case when the interest rate r is non zero, the (perfect) hedging equations become:

$$\begin{aligned} \varphi\, s_u + \psi e^{rt} &= h_u \\ \varphi\, s_d + \psi e^{rt} &= h_d \end{aligned}$$

After eliminating ψ from the initial condition $\psi = v_0 - \varphi s_0$ we get the capital gains equations:

$$\begin{aligned} v_0 e^r + \varphi(s_u - s_0 e^{rt}) &= h_u \\ v_0 e^r + \varphi(s_d - s_0 e^{rt}) &= h_d \end{aligned}$$

The balancing equations for the dual multipliers q_u, q_d are now $q_u + q_d = 1$ and $q_u(s_u - s_0 e^{rt}) + q_d(s_d - s_0 e^{rt}) = 0$ or

$$\mathbb{E}_Q S_1 = s_0 e^{rt} \tag{22}$$

whose interpretation is that q_u, q_d are probabilities under which the expected value of the stock after time t grows by e^{rt}. We call such probabilities $Q = (q_u, q_d)$ a risk neutral measure for the stock price. Their values are:

$$\begin{aligned} q_u &= \frac{s_0 e^{rt} - s_d}{s_u - s_d} \\ q_d &= \frac{s_u - s_0 e^{rt}}{s_u - s_d} \end{aligned}$$

Combining the capital gains equation we find that $v_0 e^{rt} = q_u h_u + q_d h_d$. The initial value now has to equal the **discounted value** of the final claim with respect to the risk neutral measure.

$$v_0 = \frac{q_u h_u + q_d h_d}{e^{rt}} = e^{-rt}\mathbb{E}_Q H$$

The optimal number of stock units is unchanged $\varphi = \frac{h_u - h_d}{s_u - s_d}$ and the optimal loan is given by: $\psi e^{rt} = \frac{h_d s_u - h_u s_d}{s_u - s_d}$.

In conclusion, for any interest rate r, if the future would consist only in one out of two possible states, there would exists a unique risk neutral measure and an exact hedging strategy would be possible. By charging an initial payment v_0 which equals the **discounted value** of the final claim with respect to the risk neutral measure, and investing it as indicated, the seller of any derivative product could ensure that he can pay it off without any risk. Hence the price for the derivative must be $v_0 = e^{-rt}\mathbb{E}_Q H$. Any other price would allow arbitrage as one could use the optimal hedging strategy, either buying or selling the derivative, and make guaranteed profits.

3.3 Connecting the binomial and exponential Brownian motion models

The Black Scholes formula $v_0 = \mathbb{E}^* e^{-rT} h(S_T)$ for valuing derivatives under the exponential Brownian motion model $S_T = s_0 e^{\mu T + \sigma B_T}$ by adjusting the drift of the exponent to $r - \frac{\sigma^2}{2}$ is derived under a specific continuous model, and under the asssumption of continuous rebalancing of the hedging portfolio (at no transaction costs).

At the opposite end, the binomial pricing formula is derived under the assumption of no intermediate trading, but by assuming that the stock can only move at the end of the period to one of two values.

Initially, we would guess that the prices produced by these two completely unrelated models should be quite different. However, this is not so, provided that the two values of the binomial distribution are judiciously chosen to approximate the distribution of $S_T = s_0 e^{\mu T + \sigma B_T}$. Below, we will use the approximation:

"Discrete approximation for Brownian motion": If an asset evolves as exponential Brownian motion $S_T = s_0 e^{\mu T + \sigma B_T}$, a good two value approximation for its final value is :

$$s_u, s_d = s_0 e^{\pm \sigma \sqrt{T}},$$

i.e. the exponent $\mu T + \sigma B_T$ is approximated by $Z_T = \pm \sigma \sqrt{T}$ with equal probability, independently of μ.

Note: If the Brownian motion X_T appearing in the exponent had no drift (thus $X_T = \sigma B_T$), the natural two value approximation would be of course $X_T \approx \pm \sigma \sqrt{T}$, because the two valued distribution taking the values $\pm \sigma \sqrt{T}$ with equal probability is the only symmetric two valued distribution which has the same mean and variance as the original model $X_T = \sigma B_T$. When X_T has drift, we could still keep these two values and adjust their probabilities to fit the drift by the formula $p, q = \frac{1}{2}(1 \pm \frac{\mu}{\sigma^2} D)$. However, the drift of the exponential Brownian motion model and the probabilities p, q end up thrown to the garbage anyway in the pricing process, so it is natural to disregard them from the beginning and use always the same approximation as if there was no drift!

Example 1 Let us find the price of a call option and a put option if $S_0 = 6, K = 5, \sigma = .2, r = .05$ and $t = 2$ years, both under the Black Scholes and under the binomial model based on the "discrete approximation for Brownian motion".

The table below gives the Black Scholes call value, the binomial call value, the Black Scholes put value and the binomial put value, for 24 times, starting with 8 months=2/3 year and ending with 16 years. The last two columns computed as a check $C - P + \tilde{K}$ for both models; as required by put call parity, that equals precisely $S_0 = 6$. Our answer is in the third row. Amazingly, the two models produce quite close figures, of 1.60 and 1.64, for the call. Note that in the beginning the values are quite close to the 0 volatility lower bound $S_0 - K = 1$, and as the time increases they get closer to the upper bound $S_0 = 6$ (which corresponds to ∞ volatility).

1.202	1.164	0.0382	0	6.	6.
1.41	1.413	0.0872	0.0909	6.	6.
1.598	1.643	0.1224	0.1673	6.	6.
1.771	1.84	0.1473	0.2154	6.	6.
1.933	2.013	0.1649	0.2457	6.	6.
2.084	2.17	0.1773	0.2636	6.	6.
2.226	2.313	0.1858	0.2725	6.	6.
2.362	2.445	0.1912	0.2747	6.	6.
2.49	2.567	0.1944	0.2716	6.	6.
2.613	2.682	0.1958	0.2643	6.	6.
2.731	2.789	0.1958	0.2537	6.	6.
2.843	2.889	0.1947	0.2406	6.	6.
2.951	2.984	0.1927	0.2253	6.	6.
3.055	3.073	0.1901	0.2083	6.	6.
3.154	3.157	0.1868	0.19	6.	6.
3.25	3.237	0.1832	0.1706	6.	6.
3.342	3.313	0.1792	0.1505	6.	6.
3.431	3.386	0.175	0.1297	6.	6.
3.516	3.454	0.1706	0.1085	6.	6.
3.599	3.52	0.166	0.0869	6.	6.
3.678	3.582	0.1614	0.0652	6.	6.
3.755	3.642	0.1567	0.0434	6.	6.
3.829	3.699	0.152	0.0216	6.	6.
3.901	3.753	0.1474	0.	6.	6.

2.268	2.081	2.096	2.048
2.973	2.75	2.731	2.686
3.465	3.235	3.198	3.166
3.838	3.615	3.569	3.552
4.132	3.923	3.873	3.872
4.371	4.18	4.128	4.142
4.567	4.395	4.346	4.372
4.731	4.579	4.532	4.57
4.869	4.737	4.693	4.742
4.986	4.874	4.834	4.891
5.087	4.992	4.957	5.02
5.173	5.096	5.065	5.134
5.249	5.187	5.16	5.234
5.315	5.267	5.245	5.321
5.372	5.338	5.321	5.398
5.423	5.4	5.388	5.466
5.468	5.456	5.447	5.526
5.509	5.506	5.5	5.579
5.544	5.55	5.548	5.626
5.577	5.59	5.591	5.667
5.606	5.625	5.629	5.704
5.632	5.657	5.663	5.737
5.655	5.686	5.694	5.765
5.677	5.712	5.721	5.791

Example 2 Let us find the price of the same call option under the Black Scholes and under the binomial model, if $\sigma = .4$

The table above gives first the binomial price, then two other approximations discussed in the next section, and finally the Black Scholes price, for 24 periods starting with two years and ending with 48 years. We note first that for large times, the values get very close to the upper bound S_0.

We note also that all four models produce close results; however, these are far from the previous ones obtained when we estimated the volatility to be .2. In conclusion, **the estimated volatility has a bigger impact on the price than the model we use**.

The **volatility** of a process is estimated from the equation $\sigma^2 \frac{N}{Y} = \sum_{i=1}^{N} (\log(S_{i+1} - \log S_i)^2$, where N is the total number of observations and Y is the number of trading days per year (so that $\frac{N}{Y}$ is the total time observed).

This is however only past observed volatility. The main issue of option pricing is not the model used, but the forecasting: what will be the future volatility be?

In the next section we will show how we can improve on the binomial model by allowing several intermediate "review" times at which the portfolio may be rebalanced.

3.4 The multiperiod binomial model

We will refine now the two valued approximation of the Brownian motion from the previous section to the natural situation when we observe the process a finite number of times situated at intervals $t_1, t_2, ..., t_n$, where $\sum_i t_i = t$.

At each step, we use the "discrete approximation for Brownian motion"

$$dX_{t_i} \approx \pm\sigma\sqrt{t_i}.$$

This results in the following pricing method, which is best performed organizing the computations as a tree.

The multiperiod binomial model:

1. For each $i = 1, ..., n$ we compute the number $r_i = rt_i$, the two price multipliers $u_i, d_i = e^{\pm\sigma\sqrt{t_i}}$ and the two risk neutral probabilities

$$p_i = \frac{e^{r_i} - d_i}{u_i - d_i}, q_i = \frac{u_i - e^{r_i}}{u_i - d_i}$$

2. For each branch of the tree, we compute the risk neutral probability q_w by multiplying all the probabilities along the branch, the final asset price s_w and the final claim h_w.
3. We evaluate the risk neutral expectation:

$$v_0 = e^{-rt}\mathbb{E}_Q H = e^{-rt} \sum q_w h_w$$

Exercise 3: Suppose that two "review" stages of $.75t, .25t$ are chosen for hedging the call option of example 2. Using the multiperiod binomial model, find the risk neutral probabilities at each stage and the initial value which makes exact hedging possible. **Ans:** See table 2.

Exercise 4: Redo exercise 3, if the interest rate over the first period is $r_1 = .1$ and over the second period it is $r_2 = 0$.

Conclusion: If enough review stages are chosen, the multiperiod binomial model is essentially identical to the Brownian motion model, and has the extra flexibility of allowing the introduction of additional features specific to each stage.

Note: It may be shown by induction that the multiperiod binomial method **hedges exactly any claim** (in this model, the number of possible distinct final future states is 2^n, but we also have 2^n decision variables, one for each branch of the tree)!

We explain now how perfect hedging works, by "backwards induction" (also called dynamic programming). Consider an option with expiry in n periods (for concreteness, say $n = 2$.) At expiry, "nature" can be in any of 2^n states with corresponding payoffs h_w. Our recipe for perfect hedging over one step specifies the necessary value for the hedging portfolio at time $n - 1$;

this necessary value maybe viewed as a claim at time $n-1$; to hedge this, we will need a certain value at time $n-2$, etc. In conclusion, we will need to find the "necessary" value at time t v_t; this function will be entirely determined by the value of the stock at time t, i.e. $v_t = v(S_t)$.

If we allow in our toy model more than two possible values for the future evolution of the stock price, clearly perfect hedging ceases to be possible. Also, the risk neutral measure (balancing probability) ceases to be unique (since we still have only two equations to determine it, but more variables q_w. Instead, there is a whole set of balancing probabilities, which yield different expectations for the final claim $\mathbb{E}_Q H(S_T)$. Which one should one use?

The answer to this puzzle is that all these risk neutral expectations provide reasonable initial prices, in that that they and only they eliminate the possibility of arbitrage. Furthemore, each of these potential initial prices corresponds to a certain optimization objective, which expresses the seller's and buyer's attitudes towards the "mishedge" (the hedging error). In conclusion, the set of all possible RN measures gives rise now to a **confidence band** of RN prices

$$\inf_Q \mathbb{E}_Q H(S_T) \leq \sup_Q \mathbb{E}_Q H(S_T)$$

rather than to a unique price.

Any choice within this **risk neutral band** of prices is now possible. We illustrate this in the next sections, where we show that the highest possible price coincides with the price which the seller would like to impose (so that he incurs no risk) while the lower price is the price which would eliminate the buyer's risk, while compromise attitudes like minimizing the least squares of the mishedge error lead to prices in between.

3.5 Super and sub replicating in multinomial models

The buyer and the seller of a derivative contract have different opinions about what is a fair price; the first can agree only to "subreplicating" portfolios (see below) while the other can agree only to "superreplicating" portfolios.

Consider now a market in which a stock with current value s_0 can move after one period to any of a finite set of possible values s_w. To preclude arbitrage, we need to assume that the value of the stock can go both above and below s_0. A bank sells a financial derivative which will pay h_w in the case when the value of the stock becomes s_w. The seller would like to charge the buyer a certain price $v_0^{(S)}$ which has to be enough to allow him to create a "hedging" portfolio whose value will **surely end up higher than the derivative** (thus allowing him to provide it). This is achieved by buying some number φ of stock units. The value of the hedging portfolio after one period is again given by the capital gains equation $v_0 + \varphi(s_w - s_0)$. The **seller's pricing problem** is to chose $\varphi, v_0^{(S)}$ which solve the linear programming problem

$$\begin{aligned} &\min v_0^{(S)} \quad \text{subject to} \\ v_0^{(S)} + \quad &\varphi(s_w - s_0) \geq h_w \;\; \text{for any event } w \end{aligned}$$

The seller's pricing problem expresses the seller's wish to price as cheaply as possible his product, under the constraint that the the value of the hedging portfolio at time 1 will allow him to pay the claim with no risk.

Similarly, the **buyer's pricing problem** is:

$$\max v_0^{(B)} \quad \text{subject to}$$
$$v_0^{(B)} + \varphi(s_w - s_0) \le h_w \qquad \text{for any event } w$$

The buyer's pricing problem expresses the buyer's agreement to pay for the claim as much as possible, as long as he is absolutely sure that the seller's hedging portfolio can never exceed the claim.

Exercise: Consider a one period market with three values and interest rate $r = 0$. Assume three possible future scenarios, with claim values $h_u = 9, h_m = 6, h_d = 1$, depending on whether the price of the stock goes from $s_0 = 2$ to $s_u = 3, s_m = 1.5, s_d = 1$.

a) Plot on a $S, H(S)$ graph the three possible scenarios.

b) Note that the seller's problem of finding φ, ψ so that $\psi + \varphi s_w \ge h_w$ may be interpreted as looking for the line which is above all the three points, and which has the lowest intercept over $S = s_0$. Similarly, the buyer's problem of finding φ, ψ so that $\psi + \varphi s_w \le h_w$ may be interpreted as looking for the highest line which is above all the three points and has the highest intercept over $S = s_0$.. Determine these lines, first graphically and then symbolically (i.e. find the hedging formulas for φ, ψ preferred by the seller and the buyer, respectively.

b) Find the initial prices recommended by the seller and buyer. Is the market complete, i.e. is the buyer price equal to the seller price?

c) Show that the seller and buyer's initial prices can both be expressed as expectations of the claim values with respect to certain risk neutral measures (to be determined).

Solution

a) We find graphically that the seller is concerned about the middle and upper cases, while the buyer is concerened by the extreme cases. The formulas for φ, ψ are found by plugging in the formulas from the binomial section $\varphi = \frac{h_u - h_d}{s_u - s_d}, \psi = \frac{h_d s_u - h_u s_d}{s_u - s_d}$ the respective cases.

b) Using the formula $v_0 = \varphi s_0 + \psi$ we find that the seller's price is 7 and the buyer's price is 5. The market is incomplete.

c) The two risk neutral measures of the seller/buyer, obtained by plugging the respective values in the formulas from the binomial section

$$q_u = \frac{s_0 - s_d}{s_u - s_d}, q_d = \frac{s_u - s_0}{s_u - s_d}$$

are $q_u = 1/3, q_m = 2/3, q_d = 0$ and $q_u = 1/2, q_m = 0, q_d = 1/2$.

One way out of the seller-buyer conflict by using the regression line of the points above will be given in the next section.

3.6 Choosing among several risk neutral measures **

To resolve the buyer-seller conflict which is due to the impossibility of perfect hedging, we have to introduce some specific minimization criterion for the "mishedge" (i.e. the difference between the claim and the final value of the hedging portfolio).

One possibility is to choose to minimize the expected value of the **square** of the "mishedge." Other reasonable choices are powers, exponentials, logarithms and they are referred to as **utility** functions. With the quadratic utility we are thus led to the optimization problem:

$$\min_{v_0,\varphi} \mathbb{E}(H - v_0 - \varphi(S - s_0))^2 = \sum_w p_w(h_w - v_0 - \varphi(s_w - s_0)^2 \tag{23}$$

Notes 1) In this objective we see for the first time appearing the estimated probabilities p_w of the various scenarios.

2) The least squares objective is a compromse which keeps in check both the buyer's and seller's ambitions (who have the conflicting aspirations to make the above difference positive and negative, respectively).

(23) is a classical regression problem (of the vector H on the vector $dS = S - s_0$ and on a vector of ones $\bar{1}$). The well known solution is

$$\begin{aligned} \varphi &= \frac{\text{Cov}\,(H, dS)}{Var(dS)} \\ v_0 &= \mathbb{E}H - \varphi\mathbb{E}dS. \end{aligned}$$

Letting $\mu = \mathbb{E}(dS), \sigma^2 = Var(dS)$ denote the mean and variance of the change in the stocks price (with respect to the estimated measure $P = (p_w)$, and plugging $\text{Cov}\,(H, dS) = \mathbb{E}[H(dS - \mu)]$ we find that v_0 simplifies to

$$v_0 = (1 + \frac{\mu^2}{\sigma^2})\mathbb{E}H - \frac{\mu}{\sigma^2}\mathbb{E}[H\ dS] = \sum_w h_w q_w$$

where $q_w = p_w(1 + \frac{\mu^2}{\sigma^2} - \frac{\mu}{\sigma^2}(s_w - s_0))$, which can again be interpreted as an expectation.

Theorem 3.2. *The initial value v_0 which leads to the optimal hedging of the claim in least squares sense is given by*

$$v_0 = \mathbb{E}_Q H$$

where the measure Q is given by $q_w = p_w(1 + \frac{\mu^2}{\sigma^2} - \frac{\mu}{\sigma^2}(s_w - s_0))$.

The measure Q has total mass 1 is risk neutral (but unfortunately may have negative components q_w).

Exercise: Check the last statement.

Solution It is easy to check that $\sum_w q_w = 1 + \frac{\mu^2}{\sigma^2} - \frac{\mu}{\sigma^2}\mathbb{E}[dS] = 1$.

Also, Q is risk neutral: $\sum_w q_w(s_w - s_0) = (1 + \frac{\mu^2}{\sigma^2})\mathbb{E}[dS] - \frac{\mu}{\sigma^2}\mathbb{E}([dS]^2) = \mu(1 + \frac{\mu^2}{\sigma^2} - \frac{\mu^2+\sigma^2}{\sigma^2}) = 0$

Note: It is also possible to derive the same RN measure by minimizing $\sum_w p_w \left(\frac{q_w}{p_w}\right)^2$ over the set of all RN measures.

Namely, letting $z_w = \frac{q_w}{p_w}$, where p_w, q_w are the "real world" and rsik neutral probabilities for the stock, we are looking for a solution of the "dual" problem

$$\begin{aligned}
\min_Q \mathbb{E}_P \left(\frac{dQ}{dP}\right)^2 &= \min \sum_w p_w z_w^2 \\
\sum_w p_w z_w (s_w - s_0) &= 0 \\
\sum_w p_w z_w &= 1
\end{aligned}$$

Note that if we drop the first constraint, the solution of the remaining problem would be $z_w = 1$,, i.e. $Q = P$ itself. The dual problem can thus be interpreted as trying to find a measure which is as close as possible to the observed P (in a quadratic sense) and which is also risk neutral. It was later shown by Karatzas, Lehoczky, Shreve and Xu(1991) and by Kramkov and Schachermayer(1998) that to any possible "utility" (penalty) function of the mishedge (like the quadratic here) there corresponds a unique appropriate "dual" distance between Q and P which needs to be minimized.

Exercise: Solve the dual minimization problem for z_w.

Solution: By the method of Lagrange multipliers the solution must be of the form $z_w = k_1(s_w - s_0) + k_2$; we find that $k_1 = -\frac{\mu}{\sigma^2}, k_2 = 1 + \frac{\mu^2}{\sigma^2}$ where $\mu = \mathbb{E}(S - s_0), \sigma^2 = Var(S - s_0)$.

4 Stochastic models in finance

Stochastic processes are crucial for the understanding of modern finance and insurance. They are immensely useful and have become the common language of workers in many apparently unrelated areas like finance, physics and social sciences, connected only by their common interest of untangling as time passes of the predictable from the uncertain in the unfolding of future events.

In this section we will visit briefly the most frequently used stochastic processes in mathematical finance and mention some problems where they are used. We will introduce first two general families of processes:

- Levy (or **additive**) processes, which are sums of identical independent summands, and
- **multiplicative** processes which are products of identical independent factors.

Finally, we will discuss the favorite process in mathematical finance, **exponential Brownian motion** and some applications.

4.1 Levy (additive) processes

4.1.1 Random walks

These are processes of the form

$$S_T = \sum_{t=1}^{T} X_t$$

where X_t are **i.i.d.** (independent identically distributed) random variables. In the case when $X_t = \pm 1$ the process (4.1.1) is called a **simple random walk**.

When the probabilities of going right or left are both equal to $w.p..5$ we have a **symmetric random walk** and if we allow unequal probabilities $p \neq q$ for moving right and left we have a **biased random walk.**

4.1.2 Compound Poisson processes

In continuous time, we generalize by allowing our process to jump after arbitrary intervals of time t_i. Letting N_T denote the total number of jumps which occurred in the interval $[0, T]$, we consider thus processes of the form:

$$S_T = \sum_{t=1}^{N_T} X_t$$

In order for this process to be Markovian it is necessary to assume that the interarrival times T_i are exponentially distributed, which is equivalent to the counting process N_T being a Poisson process. In this case we call the process (4.1.4) a **compound Poisson** process or a **pure jump** process.

4.1.3 Levy processes

Random walks and compound Poisson processes have both the property that there increments over disjoint time intervals are independent, with a distribution which depends only on the length of the time interval (and not on its starting point).

Levy processes are defined as the family of all processes satisfying the conditions above. Thus:

Definition 4.1. *A Levy process is any process Y_t for which:*

- *The increment $Y_t - Y_s$ of a Levy process over an interval $[s, t]$ is independent of the increment over any other time interval disjoint from $[s, t]$.*
- *The distribution of the increment $Y_{s+t} - Y_s$ is the same as that of the initial increment of time length t. i.e. $Y_t - Y_0$, independently of s.*

Briefly, we say that a Levy process has stationary, independent increments.

As noted, both ransom walks and compound Poisson processes satisfy these properties. The same is true about linear deterministic motion rt.

The exercises below will show that first moments, variances as well as cumulant generating functions (to be defined below) of Levy processes increase linearly with time (in this sense, we may think of Levy processes as "linear" random processes).

They are based on the following calculus lemma:

Lemma 4.2. *If a continuous function $f(t)$ satisfies for any s, t, the identity:*

$$f(t + s) = f(t) + f(s)$$

then $f(t)$ must be a linear function, and so $f(t) = f(1)t$.

Exercise 4.1 Show that if Y_t is a Levy process, than at any time t we have:

a) $m(t) = \mathbb{E}Y_t = t\mathbb{E}Y_1$

b) $v(t) = \operatorname{Var} Y_t = t\operatorname{Var} Y_1$

c) Establish a), b) directly (without using Lemma 1) for integer times t.

Exercise 4.2 Show that the the moment generating function of any Levy process Y_t $M(u, t) = \mathbb{E}e^{uY_t}$ satisfies the identity $M(u, t + s) = M(u, t)\ M(u, s)$. Conclude that $M(u, t)$ is of the form $M(u, t) = e^{tc(u)}$.

The function

$$c(u) = \frac{log(\mathbb{E}e^{uY_t})}{t}$$

is called **cumulant generating functional**.

Note: The cumulant generating functional of a Levy process characterizes uniquely the process (since the moment generating function does) and is typically easier to compute than the density.

In the next exercise we obtain formulas for the expectation, variance and cumulant generating functional of a compound Poisson process S_t (with bounded jumps) whose jumps' density is $f(x)$.

Exercise 4.3 a) Show that $\mathbb{E}S_1$, $\text{Var}\, S_1$ for a compound Poisson process are given by:

$$\mathbb{E}S_1 = \lambda \mathbb{E}X_1$$
$$\text{Var}\, S_1 = \lambda \mathbb{E}(X_1)^2$$

b) Compute the moment generating function $\mathbb{E}e^{\theta S_t}$ compound Poisson process S_t (assuming bounded jumps) and show that $c(\theta)$ is given by:

$$c(\theta) = \lambda(M_{X_1}(\theta) - 1) = \lambda(\int_0^\infty e^{\theta x} f(x)dx - 1) \tag{24}$$

Another simple type of continuous time Levy process is obtained by adding together a deterministic linear trend pt and a compound Poisson process $S_T = \sum_{t=1}^{N_T} X_t$.

$$Y_T = pT + S_T$$

Except for allowing both negative and positive jumps, the model (4.1.3) above is precisely the classical model of the reserves of an insurance company.

4.1.4 Application: Insurance premia

The aggregate claims process for an insurance company over a fixed time interval T is modeled by a compound Poisson process

$$S_T = \sum_{t=1}^{N_T} X_t$$

We list below a couple of proposed recipes for determining insurance premia. We denote by p the premium per year. Then , the premium per T years pT may be computed from:

1. The **mean value principle**

$$pT = (1+\theta)\mathbb{E}S_T$$

by which the company tries to cover its claims, with an extra "safety" factor of θ.

2. The **variance principle**

$$pT = \mathbb{E}S_T + \frac{\theta}{2}\text{Var}\, S_T$$

by which the company makes its extra safety charge proportional to the variance (which maybe viewed as a measure of the risks incurred). The factor θ is called risk tolerance.

3. The **exponential principle**

$$p = \frac{c(\theta)}{\theta}$$

We will see in Exercise 4 below that the exponential and variance principles are very close to each other for small θ (the variance premium and the two terms Taylor expansion around $\theta \approx 0$ of the exponential premium coincide).

Exercise 4.4 Let $c(u,t)$ denote the cumulant generating functional of a general stochastic process X_t, defined by the equation $\mathbb{E}e^{\theta X_t} = e^{c(\theta,t)}$. Derive formally the Taylor expansion

$$c(\theta,t) = \theta\ \mathbb{E}X_t + \frac{\theta^2}{2}\text{Var}\, X_t + ...$$

Conclude that the two terms Taylor approximation for the exponential premium $\frac{c(\theta)}{\theta}$ of a compound Poisson process S_t is given by:

$$\frac{c(\theta)}{\theta} = \mathbb{E}S_1 + \frac{\theta}{2}\text{Var}\, S_1 + ... = \lambda(\mathbb{E}X_1 + \frac{\theta}{2}\mathbb{E}X_1^2)$$

The importance of the expectation principle is provided by its connection to the so called "ruin probability" described in the next theorem, which will be established in a future section.

Theorem 4.3. *(Lundberg's approximation) Let $U_t = u + pt - S_t$ denote the reserves process of an insurance company with inital reserves u and premium rate p. Let $\varphi(u)$ denote the* **ruin probability***, i.e. the probability that U_t* **ever** *becomes negative*

$$\varphi(u) = \mathbb{P}_u\{U_t \leq 0,\ \text{for some } t > 0\}$$

Then, for any $\theta > 0$, choosing the premium rate by the exponential principle $p = \frac{c(\theta)}{\theta}$ ensures that the ruin probability is exponentially small with asymptotic rate decay θ, i.e.

$$\varphi(u) \approx e^{-\theta u}$$

The concern for having a large rate of decay θ and thus a small ruin probability reflects a conservative point of view: a concern with an event of small probability but catastrophic consequences.

Note: In practical computations θ is very small and thus it is alright to replace the exponential principle by its variance approximation:

$$\frac{c(\theta)}{\theta} = \lambda(\mathbb{E}X_1 + \frac{\theta}{2}\mathbb{E}X_1^2)$$

In conclusion, finding insurance premia is based on computing expectations, variances, or cumulant generating functions of a Levy process. The last principle is supported by important conservative concerns; for small θ however the variance principle which requires less estimation provides a very good approximation.

The linearity of all the above premia in time (a desirable property for a pricing principle) was ensured by linearity properties of Levy processes established in Exercises 1-3.

4.1.5 Brownian motion

The class of Levy processes includes one more types of process: Brownian motion, which maybe obtained as a limit of the symmetric and biased random walks discussed in a previous section.

Historical note:

Brownian motion was first introduced by Einstein to model the motion of pollen particles in a suspension, under the impact of collisions from electrons. In one dimension, this movement may be modeled as a sum of small jumps $\pm D$ (think of D as of the average movement of the pollen particle to the right or left), which occur after very smalll time intervals of h. In a fixed interval of time t there will be about $n = t/h$ jumps, and so the total movement will be $S_n = \sum_{i=1}^n X_i$.

Einstein realized that in order to obtain a finite limit as $h, D \to 0$ this quantities had to be related to each other. Indeed, computing the variance of S_n we find:

$$\text{Var}\, S_n = n\text{Var}\, X_1 = n\mathbb{E}X_1^2 = \frac{t}{h}D^2$$

and thus to get a finite limit for fixed t we need to assume that $D^2 \approx kh$ for some constant k. Using this relation (called Einstein scaling) Einstein was able to compute accurately the Avogadro number, which is related to k. Since k depends on the chosen units of length and time, we will mostly ignore it (take it as 1).

We assume thus the Einstein scaling $D = \sqrt{t}$ and define Brownian motion B_t as the limit when $h, D \to 0$ of the random walk described above. Brownian motion is only a mathematical idealization. In real life, we can only observe random walk. However, this continuous limit turns out to be more convenient when doing analytical computations than the "real life" random walk.

Note that as both $h, D \to 0$, the jumps of the random walk are small, but still quite large when compared to the time interval. Thus, Brownian motion is a continuous model for a sum of very frequent quite large shocks which can go equally up or down.

Definition 4.4. *Let* $S_h(t) = \sum_{i=1}^{\lfloor t/h \rfloor} X_i$ *denote a symmetric random walk with increments* $X_i = \pm D$ **with probabilities** $1/2$, *and occurring after time intervals of* h, *where* $D = \sqrt{h}$ *and* $\lfloor t/h \rfloor$ *denotes integer part.*

Standard Brownian motion *is the limit of symmetric random walks:*

$$B_t = \lim_{h \to 0} S_h(t)$$

Theorem 4.5. *Standard Brownian motion $B(t)$ is characterized by the following three properties:*

1. *$B(t)$ has a Gaussian distribution with mean 0 and variance t. Its density is thus: $\frac{\exp(-\frac{x^2}{2\,t})}{\sqrt{2\pi\, t}}$.*
2. *It is a Levy process: thus it has stationary increments (meaning that $B(t) - B(s)$ has the same distribution as $B(t-s)$), and increments over disjoint time intervals are independent.*
3. *Brownian motion has continuous paths*

Proof of 1:

1. The Gaussian distribution is a consequence of B_t being a sum of infinitely many small shocks (by the central limit theorem which states that sums of i.i.d's have a nearly Gaussian distribution). Formally, we need to check that the first and second moments of the process $S_h(t)$ converge to those of the Gaussian distribution prescribed ($\frac{\exp(-\frac{x^2}{2\,t})}{\sqrt{2\pi\, t}}$), which has mean 0 and variance=second moment t. Indeed, it is easy to check that the mean of $S_h(t) = 0$ equals 0 and the variance of $S_h(t)$ is $n\; D^2 = \lfloor t/h \rfloor\; h \to t$.

Thus, by the central limit theorem it follows that the limiting distribution of $S_h(t)$ has to be Gaussian with mean 0 and variance t.

Property 2 follows immediately from the corresponding property of the random walks. Property 3 is very hard to establish and beyond the scope of these notes.

Exercise 4.5 a) Compute the moment generating function $\mathbb{E}e^{uN}$ of the standard normal random variable.

b) Compute the moment generating function $\mathbb{E}e^{uX}$ of normal random variable with mean 0 and standard deviation σ. **Hint:** X may be represented as $X = \sigma N$.

c) Find $\mathbb{E}e^{u\sigma B_t}$ if B_t is stadard Brownian motion.

Exercise 4.6 If $B(t)$ is standard Brownian motion, find $\mathbb{E}B(s)B(t)$ if $s < t$. Hint: Write $B(t) = B(s) + [B(t) - B(s)]$, and use the independence of $B(s)$ of the increment $B(t) - B(s)$.

4.1.6 Brownian motion with drift

Adding together a deterministic trend μt and an "amplified" Brownian motion $\sigma B(t)$ yields the process

$$B_{\mu,\sigma^2}(t) = \mu\, t + \sigma B(t)$$

called **Brownian motion with drift**. Note that its mean and variance at time t are respectively:

$$\mathbb{E}B_{\mu,\sigma^2}(t) = \mu\, t$$
$$\mathrm{Var}\, B_{\mu,\sigma^2}(t) = \sigma^2\, t.$$

The density is thus

$$\frac{\exp\left(-\frac{(x-gt)^2}{2\sigma^2\, t}\right)}{\sqrt{2\pi\sigma^2\, t}}.$$

Exercise 4.7 Find the moment generating function of Brownian motion with drift $\mathbb{E}e^{uB_{g,\sigma}(t)}$ and its cumulant generating function .

The drift parameter μ will also be called growth parameter; this name is suggested by the theorem below, a consequence of the law of large numbers, which shows that for large times the drift dominates the oscillation part of the Brownian motion with drift.

Theorem 4.6. *The long term behavior of Brownian motion with drift is:*

$$\lim_{t\to\infty} \frac{B_{\mu,\sigma^2}(t)}{t} = \mu$$

By this result, we may think of Brownian motion with drift as a process which oscillates wildly on a short time scale (due to σB_t) around a long term trend of μt.

The "real life" analog of Brownian motion is the **biased random walk with vanishing bias** defined below.

We call a process $S_h(t)$ a "biased random walk with vanishing bias" if it is of the form: $S_h(t) = \sum_{i=1}^{n} Z_i$ with $X_i = \pm D$ with probabilities $p, q = 1/2 \pm \frac{\mu}{2\sigma^2}\, D$, $n = \lfloor t/h \rfloor$, and $D, h \approx 0$ satisfying the scaling relation $D^2 = \sigma^2 h$.

We will check now that the first and second moments of the process $S_h(t)$ defined above converge to those of the Brownian motion $B_{\mu,\sigma^2}(t)$. Indeed, the mean of X_1 is $D\,(p-q) = D\frac{\mu}{\sigma^2} = h\, \mu$ and thus the mean of $S_h(t)$ is $n\, h\, \mu \to t\, \mu$. Similarly, the variance of X_1 is $D^2 - h^2\mu^2 \approx h\, \sigma^2$ and so the variance of $S_n(t)$ is $n\, h\, \sigma^2 \to t\, \sigma^2$.

Theorem 4.7. *The limit of the biased random walk with vanishing bias $S_h(t)$ defined by the parameters μ, σ is the Brownian motion with drift corresponding to those parameters.*

Note: Brownian motions are actually easier to work with than their discrete counterparts, as far as analytic manipulations are concerned . We will see in a later section that while working with discrete random walks requires computing sums and solving difference equations, working with Brownian motion requires computing integrals and solving differential equations, which are somewhat easier analyically. For example, the distribution of symmetric random walk at a fixed time is given by some complicated sums, while the distribution of Brownian motion at a fixed time has a simpler formula (provided by the Gaussian density).

4.1.7 Classification of Levy processes

We quote now an important theorem whose proof is beyond the scope of these notes, which shows that all Levy processes are made up of three components: the deterministic trend, a compound Poisson part and Brownian motion.

Theorem 4.8. Decomposition of Levy processes *Any Levy process may be decomposed as a sum of three parts.*

$$Y_t = pt + S(t) + \sigma B(t)$$

where $B(t)$ is standard Brownian motion. Strictly speaking, the part $S(t)$ maybe more general than a compound Poisson process, in the sense that it may allow jumps to occur with infinite frequency in which case it is called **pure jump Levy process**

Note that a Levy model is characterized by three scalar parameters: λ, p, σ and a "function parameter", the distribution $F(x)$ of the jumps, and is as such much richer than a Brownian motion model, which has only two parameters: the deterministic trend p and the volatility σ.

Unlike Brownian motion, Levy processes don't usually have simple density formulas, but they do have simple cumulant generating functions. This is illustrated in the exercise below where we compute the cumulant generating functional for a general Levy processes.

Exercise 4.8 Using the decomposition theorem for Levy processes, show that the cumulant generating functional for a Levy process whose decomposition is $Y_t = pt + \sigma B_t + S_t$ is given by:

$$c(u) = pu + \frac{\sigma^2}{2}u^2 + \lambda \int (e^{uz} - 1) f(z) dz$$

Note: Despite the fact that Brownian motion, compound Poisson processes and deterministic motion are so different at first sight, most of their properties can be expressed in a unified way by using the cumulant generating function $c(u)$.

4.2 Multiplicative (exponential) processes

Random walks are **additive processes**, by which we mean that they satisfy a recursive formula

$$X_{t+1} = X_t + Z_{t+1}$$

In finance, in keeping with the mechanism of compounding of interest, we are also intersted in **multiplicative processess** satisfying a recursive formula:

$$S_{t+1} = S_t Z_{t+1}$$

where Y_i are independent identically distributed random variables.

Thus, a multiplicative process is a product

$$S_t = \prod_{i=1}^{t} Z_i$$

of positive factors. The decomposition above suggests the following result

Lemma 4.9. *A process in discrete or continuous time is multiplicative iff its logarithm is additive.*

Thus, multiplicative processes are precisely the exponentials of Levy processes.

Example: Exponential Brownian motion (EBM) is the process

$$S_t = S_0 e^{pt+\sigma B_t}$$

The birth of modern mathematical finance may be traced to the adoption of the EBM model by Samuelson, around 1960 (previous attempts around 1900 by Bachelier to model asset evolution as Brownian motions with drift has a lesser impact).

A class of processes of great importance in finance are the so called **risk neutral** processes. Typically, we assume the existence of a deterministically increasing "riskless" investment which yields a fixed interest rate r.

Definition: A process S_t is called risk neutral if its expectation increases exponentially at rate r:

$$\mathbb{E}S_t = e^{rt} S_0$$

(i.e., the expected increase equals that of the riskless investment).

Exercise 1.9 below shows that the study of risk neutral processes may be reduced to that of the special case (obtained when $r = 0$) of processes with constant expectation. These processes figure also prominently in the theory of gambling, where they received the name of **martingales** (from a certain gambling strategy).

Definition A **martingale** is process which has constant expectation

$$\mathbb{E}S_{t+s} = \mathbb{E}S_t = S_0$$

Informally, martingales are processes of "balanced" evolution, in the sense that their expected increase and expected decrease counterbalance each other on average.

In discrete time, it is easy to see that an additive process is a martingale iff its increments have mean 0 and a multiplicative process is a martingale iff its factors have mean 1.

Exercise 4.9 A process S_t is risk neutral iff the "discounted" process

$$\tilde{S}_t = e^{-rt} S_t$$

is a martingale.

Note: The expectation of the discounted process $\mathbb{E}\tilde{S}_t = \mathbb{E}e^{-rt}S_t$ plays a crucial role in all financial considerations and is called **present value**.

Exercise 4.10

1. Find a necessary condition for a Levy process to be risk neutral.
2. Find a necessary condition for a Levy process to be a martingale.
3. Show that an exponential Levy process is risk neutral iff it satisfies the equation
$$c(1) = r$$
4. What does this condition become in the case of the exponential of Brownian motion with drift?
5. Find a necessary condition for an exponential Levy process to be a martingale.
6. What does this condition become in the case of the exponential of Brownian motion with drift?

It turns out that most of the time in mathematical finance one needs only to work with risk neutral processes and martingales.

In the following subsection we discuss the favourite model in mathematical finance: exponential Brownian motion, which is the exponential of a Brownian motion with drift.

4.2.1 Exponential Brownian motion

A process $S(t)$ is called exponential (geometric) Brownian motion if it is of the form

$$S_t = S_0 \exp\left(g\, t + \sigma B_t\right)$$

Exponential Brownian motion became in the seventies the preferred model for the evolution of stocks. The reason is that the discrete evolution of assets is most naturally modeled as a multiplicative process $S_n(t) = S_0 \prod^n \exp Y_i$ and in the continuous time limit this converges to geometric Brownian motion.

The parameter σ is called volatility, and the parameter g is called sometimes drift and sometimes "growth rate", because it provides (by the law of large numbers) the path behavior of the logarithm of the process. For example, note that the law of large numbers implies that for $g > 0$ we must have $\lim_{t\to\infty} S_t = \infty$, and for for $g < 0$ we must have $\lim_{t\to\infty} S_t = 0$. Thus, the sign of g determines the limiting behavior of the asset.

The density of exponential brownian motion has the peculiarity that it may exhibit quite large separation between the "mode" and the "expectation", as illustrated in the figure below. Thus, the most probable values (near the mode) are often quite smaller than the expectation. It turns out (as a consequence of the law of large numbers) that for t large, GBM is well approximated by e^{gt}, which turns out to equal also the median of the process. We will show in an exercise below that the expectation of geometric Brownian motion is given by $e^{(g+\sigma^2/2)t}$, the mode is only $e^{(g-\sigma^2/2)t}$, with the median e^{gt} in between.

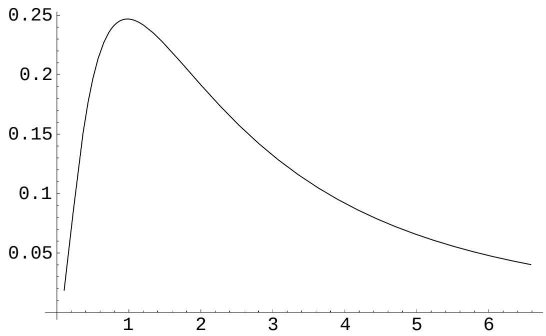

Figure 4: Density of "GBM 20 year returns": mode=.98, growth rate=2.66 , mean =4.4

As t gets large, these three numbers can get arbitrarily far away of each other. For example if $g < 0$ but $g + \frac{\sigma^2}{2} > 0$ for a certain investment, then the expectation will get arbitrarily large for large t and so the investment has "bright expectations on the average". However, the negative drift implies that the median is smaller than 1 and in fact approaches 0, and thus the investment will eventually go to 0 and thus the "bright expectations" will almost never be realized. This paradox is explained by the fact that even though geometric Brownian motion with $g < 0$ will converge to 0 in 99% of the cases, in the remaining small probablity case that it doesn't it may become so huge that the expectation, which is the average of all cases, can be huge. We have thus a process with huge expectation which almost surely goes to 0!

Exercise 4.11

1. Compute the expectation of geometric Brownian motion $\mathbb{E}S_t$. Show that GBM has constant expectation iff $g = \frac{-\sigma^2}{2}$.
2. Find $\mathbb{P}\{S_t \leq x\}$.
3. Write down the density of geometric Brownian motion if $S_0 = 1$ and compute its mode.
4. Find the median of S_t. Supposing that S_t models the evolution of a stock price, comment on whether this stock would be a good investment in case the parameters satisfy $-\frac{\sigma^2}{2} < g < 0$.
5. Show that geometric Brownian motion is a martingale iff $g = \frac{-\sigma^2}{2}$.

We discuss now some particular cases of geometric Brownian motion.

Example 1: Standard Geometric Brownian motion

This is the process $S(t) = \exp B(t)$, obtained when $\mu = 0$ and $\sigma = 1$. Note that $S(t)$ will be larger (smaller) than 1 iff the Brownian moton $B(t)$ is positive (negative). The

Brownian motion has equal probability of being positive or negative. However, this doesn't necessarily mean it is crossing the x axis all the time. If you simulate it, you will notice that each simulation usually takes off, either upward, or downward! So, a standard geometric Brownian motion stock will end up either as a marvelous investment, or as a disaster, with equal probability. We will call such a stock a potential opportunity!

Note that the expectation of this process (which may be computed by completing the square) is $\exp \frac{t}{2}$, quickly increasing to ∞ with time. This means for say 100 investors who invested in these "potential" opportunities, the average of the fortunes of the 50 winners with those of the 50 losers is heavily on the positive side. This is a well known effect of compounding geometrically over long times. The result of starting with 1000 pounds and keeping halving your fortune, say 10 times, are much less spectacular than those of doubling it 10 times!

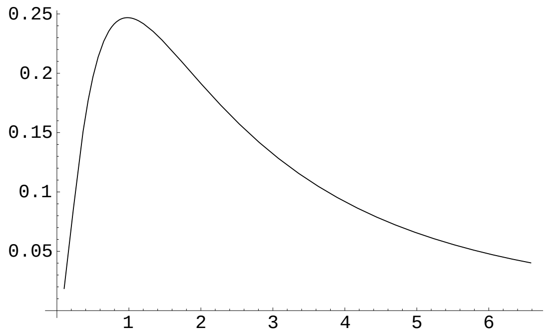

Figure 5: Density of "standard GBM 20 year returns": mode=.36, growth rate=1 , mean =1.6

We will see that in the presence of several "potential" opportunity stocks, judicious investing (continuous rebalancing) can lead to fortune!

Example 2: Exponential Martingales ("Fair" disasters)

In order to be a martingale, geometric Brownian motion has to have negative drift. The law of large numbers implies then that $B_{\mu,\sigma^2}(t) \approx \mu\, t \approx -\infty$ and so the geometric Brownian motion will converge (almost surely) to 0. Thus, a stock distributed as an exponential martingale is a sure disaster. So, what happened to "fairness"? This time it means that the one out of 100 holders of such stocks who wins wins enough to counterbalance the losses of the others.

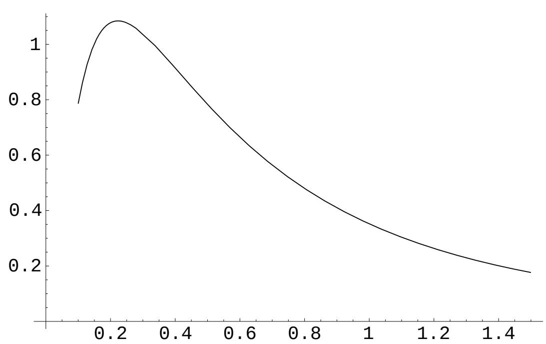

Figure 6: Density of "Martingale GBM 20 year returns": mode=.22, growth rate=.60 , mean =1

This model turns out to be important in theoretical finance.

4.3 Application: European financial derivatives

Financial derivatives (or options) are one of the most important products traded nowadays at stock exchanges.

They are contracts which grant their holder the right to receive a future payment, whose size depends in its turn on the evolution of a "primary" underlying asset (for example a stock). Thus, an option contract may result in several possible outcomes at the contracts' expiration time, depending on the evolution of the "primary" stock.

For example, a **call option**, one of the most used derivatives, gives at the expiration time T the right to either buy a primary asset S_T at a predecided **exercise price** K in case $S_T > K$, or do nothing if the price of an underlying asset fello below, i.e $S_T \leq K$. The two cases may be put together in one formula for the final payoff:

$$(S_T - K)_+.$$

Other financial derivatives are defined by applying different (almost arbitrary) payoff functions $h(S_T)$ to the final value of the underlying asset price S_T. Some examples of the most traded derivative contracts are:

- **Call options** with payoff $(S_T - K)_+$
- **Put options** with payoff $(K - S_T)_+$
- **Binary options** with payoff $1_{\{S_T \geq K\}}$
- **Spread options** with payoff $1_{\{L \leq S_T \leq K\}}$.

Note that the buyer of a call option or binary option is betting on the future price of the stock ending above K, the buyer of a put option is betting on the price ending below K, and the buyer of a spread option is betting on a precise interval for the final prize. In principle, any arbitrary function $H_T = h(S_T)$ can be used as basis for a traded option (and some are!). Thus, any function devoid of any economic interpretation, for example $Sin(Log(1 + S_T))$ could represent a valid financial derivative, which, depending on the mathematical sophistication of your local exchange, might be or not traded publicly!

The fundamental question about financial derivatives is how they should be priced: how much should one pay today for the right to receive a certain random payment in the future? The natural answer, namely forecasting the primary asset's distribution, and then taking expectation, to be illustrated in the exercises below, will turn out actually to be wrong, "but not by far". The correct answer will be provided in the following section.

A speculator who has a certain model (opinion) of the evolution of a stock would estimate the current value of his expected benefits from holding an option with final payoff $h(S_T)$ by computing the expected discounted payoff

$$\mathbb{E}e^{-rT}h(S_T)$$

also known as expected present value. Some examples follow:

Exercise 4.12 Compute the expected discounted payoff of a binary option with payoff $1_{\{S_T \geq K\}}$ under the general exponential Brownian motion model.

Exercise 4.13 Compute the expected discounted payoff of an "asset or nothing" option with payoff $S_T 1_{\{S_T \geq K\}}$ under the general exponential Brownian motion model.

Exercise 4.14 Compute the expected discounted payoff of a call option $c(S_0, K) = E(S_T - K)_+$ under the general exponential Brownian motion model.

We mention now an insurance problem of setting stop-loss reinsurance premia, which turns out formally to be almost identical to that of pricing call options, the only difference being in the different models (multiplicative versus additive) adopted.

Definition 4.10. *The stop-loss reinsurance method is a contract by which the reinsurer takes upon himself to cover the excess over a fixed amount K. Thus, letting $Y_T = \sum_{t=1}^{N_T} X_t$ denote the total claims process, the reinsured coverage is for $(Y_T - K)_+$ and thus the "expectation" reinsurance premium, computed with no loading, is*

$$\mathbb{E}(Y_T - K)_+.$$

To compute a "risk adjusted" reinsurance premium, we would need to estimate also the variance or the cumulant generating function of $(Y_T - K)_+$.

We will explain in the next section that the speculator value for an option is a very poor basis for pricing them. (It would work allright if life was eternal and we would meet the same circumstances again and again, in which case the law of large numbers would become relevant and thinks would break even "on the average".)

A better understanding of how financial derivatives should be priced was only achieved recently when researchers added to the picture the "missing link": the fact that option contracts are actively "hedged". This is a process by which option sellers insure their products, by switching money between the primary asset and some "riskless" investment. This led to a recipe called **risk neutral valuation**, to be discussed in the next section.

4.3.1 Risk neutral valuation

Risk neutral valuation involves typically assuming an exponential Brownian motion model $S_t = S_0 e^{g\, t + \sigma B_t}$, then replacing the growth parameter by $r - \frac{\sigma^2}{2}$. The new exponential Brownian motiondistribution obtained, to be denoted by S_t^* will then be used in computing the options value as:

$$v_0 = \mathbb{E}\, e^{-rT} h(S_T^*) \qquad \textbf{RN valuation}$$

Note that the growth parameter ends up being ignored altogether. Thus, ignoring for the moment the unexpected modification of the drift, risk neutral valuation is just usual computation of expectations, except that it is only applied to exponential Brownian motions with $g = r - \frac{\sigma^2}{2}$, which are risk neutral (by exercise 1.10).

More generally, for other asset models S_t (for example exponential Levy motions) risk neutral valuation consists in "modifying" somehow the estimated model S_t into a modified model S_t^* which is risk neutral, and then in computing the expected present value $\mathbb{E}e^{-rT}h(S_T^*)$ with respect to this new model. For example, we have found in Exercise 1.10 that exponential Levy motion is risk neutral only when $c(1) = r$. The risk neutral valuation recipe implies that only exponential Levy models satisfying this condition can be used in pricing.

We ask the reader to take for now risk neutral valuation as a "cook book recipe". In the following two subsections however we will sketch briefly the reasons for its use and discuss some of its consequences which look at first quite paradoxical.

Exercise 4.15 Black Scholes formula

Show that the risk neutral value of a call option under the exponential Brownian motion model is given by

$$v_0 = S_0 \Phi\left(\frac{\log\left(\frac{S_0}{\tilde{K_T}}\right) + \frac{V_T}{2}}{\sqrt{V_T}}\right) - \tilde{K_T}\Phi\left(\frac{\log\left(\frac{S_0}{\tilde{K_T}}\right) - \frac{V_T}{2}}{\sqrt{V_T}}\right) \tag{25}$$

where $\tilde{K_T} = Ke^{-rT}$ is the current value of the final exercise price and $V_T = \sigma^2 T$ is the total remaining volatility.

Note: We will also be interested in the "value" $V(t, S_t)$ provided by the the Black-Scholes formula applied at time t. For this, we replace S_0 by S_t, and T by the remaining time $T - t$, obtaining

$$V(t, S_t) = S_t \Phi\left(\frac{\log\left(\frac{S_t}{K_t}\right) + \frac{V_t}{2}}{\sqrt{V_t}}\right) - K_t \Phi\left(\frac{\log\left(\frac{S_t}{K_t}\right) - \frac{V_t}{2}}{\sqrt{V_t}}\right) = S_t \Phi(L_t) - K_t \Phi(l_t) \tag{26}$$

where $K_t = Ke^{-r(T-t)}$ is the discounted value at time t of the final exercise price and $V_t = \sigma^2(T - t)$ is the remaining total volatility at time t.

The fact that we may talk about the "value" of certain contracts at intermediate times between their initiation and their expiration is a consequence of the fact that certain contracts, including options, may be sold and bought also at intermediate times.

Another model in which the risk neutral measure is unique is the Cox-Ross-Rubinstein model in which the assets are assumed to be able to move after discrete time steps of $t = 1$ to one of only two values s_u (up) or s_d (down).

Exercise 4.16 Find the risk neutral probabilities for the Cox-Ross-Rubinstein model when $r = 0$.

4.3.2 Reasons for using risk neutral valuation

The use of risk neutral valuation is supported by two findings:

1. **A. Eliminating arbitrage**

   It may be shown that using non risk neutral measures for valuation leads to "arbitrages" which are trading strategies which reap infinite profits. Some examples of arbitrages are given in the section on the Cox-Ross-Rubinstein model.

2. **B. A recipe for hedging**

   Risk neutral valuation provides an answer to the problem of how to optimally hedge claims, under certain idealized conditions.

We will explain now the second point. To emphasize the ideas, we will assume that $r = 0$ (ensuring that the value of loans is constant).

Definition: A hedging portfolio is a combination of a number φ_t of stock units and a cash investment (or loan) ψ_t with total value

$$V_t = \varphi_t S_t + \psi_t$$

which is maintained by the seller as an insurance against the claim. This portfolio is started by charging the value

$$v_0 = \varphi_0 s_0 + \psi_0$$

to the buyer. The intention is that at the expiration time T the total value V_T of the hedging portfolio should be "as close" as possible to the value of the claim H_T. If equality holds for any possible evolution of the stock, we say that the option has been hedged exactly.

is the price to be charged to the buyer. Usually the cash investment ψ is negative, and is thus a loan; it allows the seller to hold a larger number of stock units than could have been held without using it.

The theorem below explains the importance of risk neutral valuation in complete markets, which are markets for which a unique risk neutral measure exists.

Theorem 4.11. Fundamental theorem of derivatives pricing in complete markets

a) *In a complete market in which a unique risk neutral measure denoted by $\mathbb{E}^*$ exists, arbitrage (the possiblity of unbounded profits) may be avoided iff the initial value charged for a future claim $H_T = f(S_T)$ is*

$$v_0 = e^{-r\,T}\mathbb{E}^* H_T$$

b) *If the hedger maintains at any time $t < T$ a hedging portfolio with total value*

$$V_t = V(t, S_t) = e^{-r\,(T-t)}\mathbb{E}^*[H_T/S_t]$$

and containing $\Delta_t = \frac{\partial V(t,S_t)}{\partial S_t}$ units of stock (and thus a cash investment of $\psi_t = V_t - \Delta_t S_t$), then the option will be hedged exactly under the "idealized" conditions of the Black Scoles market enumerated below.

The assumptions of the Black Scoles market are:

1. Equal rate of lending and borrowing
2. Unrestricted possibility of short selling (thus, a "bad" stock can be as good as a "good" stock, since we can shortsell it in any amount we want).
3. Possibility to trade continuously, without transaction costs.
4. An exponential geometric Brownian motion model with known future volatility.

We illustrate now part b) of the fundamental theorem of derivative pricing as applied to the forward contract.

Example 1: Hedging forwards

Consider a forward, which is a contract to deliver a stock at some time T in the future; the payment is settled however at time $t = 0$. One possible candidate for premium would be $v_0 = \mathbb{E}e^{-rT}S_T$, where $\mathbb{E}$ is expectation with respect to some estimated statistical model. By the law of large numbers, this would work alright in the long run for the seller, provided the estimated model is correct. Sometimes the seller would win and sometimes they would lose, and this would be kind of a "financial roulette" for high level bank executives.

However, this entertaining roulette need only be played in practice by the buyers, since a much more sensible strategy is available the sellers. The forward is the only option for which the hedging strategy is obvious, without using risk neutral valuation: by charging a premium S_0, the seller can buy the stock at time 0 and keep it ready for delivery until the end and thus fulfil their obligation at time T whatever the price then. By hedging this way, they have eliminated any risk on their part! Clearly, whenever hedging is possible, the right price for an option should be the initial expense necessary to set up the replicating portfolio, disregarding any possible statistical expectations $\mathbb{E}S_T$ we might have of the future. (Another argument in the favor of abandoning conjectured expectations is that if someone has strong feelings or insider info about the way S_t will evolve, he might as well buy the stock itself.)

The exercise below computes the "speculator value" and the risk neutral value of the forward and checks that the hedging strategy provided by the fundamental theorem coincides with the "buy and hold" described above.

Exercise 4.17 The "speculator" value of a forward contract is given by the expected present value of the asset $\mathbb{E}e^{-rT}S_T$. Assuming $r = 0$, find the speculator value of a forward if:

a) The speculator believes that the stock price whose initial price is a will follow a Brownian motion with drift g and volatility σ

b) The speculator believes that the price follows the exponential of the Brownian motion $S_t = a\ \exp(B_t)$ with the same parameters as before (i.e., a geometric Brownian motion with growth rate g and volatility σ)?

c) The risk neutral value of the forward contract is given by the same expectation, $\mathbb{E}^* e^{-rT} S_T$, but taken under the "closest" risk neutral measure $\mathbb{E}^*$. What is the closest risk neutral measure, and what is the risk neutral value of the forward for geometric Brownian motion?

d) Find the hedging portfolio prescribed by the fundamental theorem of derivative pricing for the forward and check that it coincides with the simple "buy and hold" strategy.

Example 2: Hedging call options

Stop loss hedging Before examining the "Black Scholes" hedging proposed by the fundamental theorem of derivative pricing , we will discuss the simplest possible hedging coming to mind, further simplified by assuming $r = 0$. This strategy, called "stop loss", consists in keeping a stock unit and a loan of K whenever the price S_t is above K and liquidating both when it gets below. Implementing this strategy requires an initial investment of $(S_0 - K)_+$. The first suspicious thing to notice about this strategy is that "out of money" call options would have 0 price. The astute buyer would then get a lot (zillions)! Since one of a zillion options is bound to get "in the money", the astute buyer would realize a profit for nothing (an "arbitrage").

Two conditions are required for the stop loss strategy to work:

1) Continuous monitoring; in discrete time the stop loss leads to "lateness" losses, since whenever you try to sell the stock **after** it moved below K, or when you try to buy it again **after** it moves above K you are bound to lose a bit.

2) The second condition is considerably less obvious: it requires that $\sigma = 0$ (a model with no Brownian oscillations), since it is known that after reaching any level, Brownian motion will cross that level again an infinite number of times in any neighborhood, no matter how small, of the first crossing.

We will see however that the optimal strategy prescribed by the fundamental theorem of derivative pricing reduces precisely to "stop loss" when $\sigma = 0$. Furthemore, the stop loss would be the optimal hedging strategy in any "smooth" differentiable continuous time market (which contained thus no Brownian motion).

Since this would lead to arbitrage, it is apparent that there is something terribly wrong about the assumption of a smooth continuous time stock market (in which people could get rich by buying 0 cost products). The "wiggliness" of Brownian motion is thus absolutely necessary in a theory of asset pricing.

fundamental theorem of derivative pricing hedging Let now $V(t, S_t)$ denote the value of the Black-Scholes formula with S_0 replaced by S_t, and T replaced by the remaining time $T - t$, i.e.

$$V(t, S_t) = S_t \Phi\left(\frac{\log\left(\frac{S_t}{K_t}\right) + \frac{V_t}{2}}{\sqrt{V_t}}\right) - K_t \Phi\left(\frac{\log\left(\frac{S_t}{K_t}\right) - \frac{V_t}{2}}{\sqrt{V_t}}\right) = S_t \Phi(L_t) - K_t \Phi(l_t) \tag{27}$$

with $K_t = K e^{-r(T-t)}$ being the value at time t of the final exercise price and $V_t = \sigma^2 (T - t)$ being the remaining total volatility at time t.

Using the exercise below, it is possible to show that $\frac{\partial V(t,S_t)}{\partial S_t} = \Phi(L_t)$.

Thus, by the fundamental theorem of derivative pricing the optimal hedge consists in holdin $\Delta_t = \Phi(L_t)$ units of stock and a cash investment of $\psi_t = V_t - \Delta_t S_t = -K_t \Phi(l_t)$.

In conclusion, $\Phi(L_t)$ represents the proportion of a stock unit which the hedger should hold at time t; it will end up at 1 or 0 depending on whether the option ends in or out of the money, i.e. on whether S_T ends up biger or smaller than K (note that since the denominator $V_T = 0$, $\Phi(\frac{\log \frac{S_T}{K} - \frac{V_T}{2}}{\sqrt{V_t}})$ is 1 or 0 depending on whether S_T ends up biger or smaller than K).

Similarly, $\Phi(l_t)$ represents the proportion of the exercise price loaned by the hedger at time t in order to be able to hold more stock; it will also end up at 1 or 0 depending on whether the option ends in or out of the money.

Exercise 4.18 ** Show that the Black Scholes value $V(t, S_t)$ of the call option satisfies

$$\frac{\partial V(t, S_t)}{\partial S_t} = \Phi(L_t).$$

Exercise 4.19 ** Show that the Black-Scholes value is an increasing function of the total variablity V (or of σ) and that it reduces to $(S_t - K)_+$ when V (or σ, or the remaining time $T - t$) becomes 0.

Conclusion: The stop loss value $(S_t - K)_+$ was of course always known to traders, and they knew equally well that uncertainty about the market increases the initial investment necessary for hedging options. The importance of the Black Scholes consists precisely in that it quantified that intuitive feeling by a precise formula dpending on a single parameter V, which is supposed to encapsulate all the future uncertainty in the market.

Note: While risk neutral valuation does not force the use of the exponential Brownian motionmodel, this model is considerably more convenient, as explained next.

C. The reason for using exponential Brownian motion

Given an estimated exponential Brownian model S_t with parameters μ, σ, or an estimated exponential Levy model with parameters λ, F, p, σ, an important question is how to choose the "right" risk neutral model (measure) for pricing. Intuitively, we should chose a risk neutral model as close as possible to the estimated one. However, the precise answer to this question is more complicated. First, we may only use measures which are **equivalent** to the original measure; this is a rather sophisticated restriction (it means that the events on which the two measures have positive probability must coincide) which we will ignore.

However, even after this restriction, it turns out typically that the set of risk neutral measures available for use is infinite. Simple examples of this nonuniqueness are presented in the section on the Cox-Ross-Rubinstein model, in which uniqueness occurs only if we restrict the number of possible future scenarios after each discrete time step to two. In the continuous setup, uniqueness exists only in the Brownian case.

Theorem 4.12. *If the estimated exponential Brownian model for an asset has parameters*

g, σ, then the unique risk neutral model ("equivalent" to the original model) is obtained by keeping σ and modifying the value of the drift to $r - \sigma^2/2$.

In cases when several risk neutral measures are available, we are only able to choose one after the specification of a "loss" function for measuring closeness. Examples are provided in the section on the Cox-Ross-Rubinstein model.

4.3.3 A paradox concerning risk neutral valuation

We note now that adopting the risk neutral valuation principle for the exponential Brownian motion model leads to paradoxical conclusions. This valuation recipe **ignores the estimated long run drift** g of the asset's log-returns and **replaces it instead with the value** $g = r - \frac{\sigma^2}{2}$ (which makes the process risk neutral).

The apparent paradox in this completely counterintuitive recipe (how can it be safe to ignore whether a stock goes up or down?) is explained by the fact that risk neutral valuation, which is based on optimal hedging, is a measure of how much initial capital the option seller needs initially to allow him enough leeway to eliminate all risk caused by later fluctuations. While this initial capital does depend strongly on the assets' volatility, due to the completely unrealistic assumptions of the Black Scholes market, it turns out that the handling of stocks which increase on the average is similar to that of stocks which decrease (more precisely, the long run trend does not affect the amount of initial capital needed).

The assumptions of the Black Scholes market are completely unrealistic for small traders; however, they are a reasonable enough approximation of the reality of big traders. This lead to the use of risk neutral valuation theory as a rough guideline for pricing.

Note: If stop loss reinsurance contracts could be traded on the exchanges under conditions similar to those described above, then their premiums could be computed (or approximated) by risk neutral valuation. It is the possibility of continuous trading which has shifted the accent from the statistical estimation and "speculator expectations" used in traditional actuarial science to the hedging optimization and "risk neutral" expectations used by investment banks.

Finding a risk neutralized process for processes other than exponential Brownian motionis considerably more complicated; the answer depends on specifying a utility function for the client (a notoriously difficcult task). One simple convenient answer available for the exponential Levy model which bypasses the utility issue is given in the next section.

4.3.4 Escher transform and valuation **

We describe now a RN valuation method which works especially well for exponential Levy processes, i.e. processes modeled as $S_t = e^{Y_t}$, where Y_t is a Levy process.

The idea is to modify the original process by multiplying its density $f(x,t)$ with an exponential factor $e^{\theta x - k}$, where θ is chosen to ensure risk neutrality. Note that the constant k has to be chosen so that the weigted density integrates to 1; this results in $k = t\ c(\theta)$, so choosing θ determines k. The resulting weighted measure is called an Escher transform.

Exercise 4.20 For a given geometric Brownian motion, determine θ so that the exponentially weighted process is risk neutral and show that the resulting Escher transformed process is a geometric Brownian motion with the drift modified to the value $r - \frac{\sigma^2}{2}$.

Since Levy processes don't usually have simple density formulas, but they do have simple cumulant generating functions, it is easier to describe the Escher transform via its effect on the cumulant generating function.

We note first that the Escher transform amounts to computing expectations of an arbitrary function $h(X_t)$ via:

$$E^{\theta} h(X_t) = \mathbb{E} e^{\theta X_t - tc(\theta)} h(X_t) dt$$

From this formula we find that the cumulant generating function of the transformed process is given by

$$c^{\theta}(u) = c(u+\theta) - \c{}(\theta)$$

We recall from Exercise 11 that a Levy process is RN iff its cumulant generating function satisfies $c(1) = r$. It follows that a Escher transformed Levy process is RN iff the parameter θ satisfies the

$$c_{\theta}(1) = c(1+\theta) - c(\theta) = r. \tag{28}$$

Example: $Y_t = u + pt - \sum_i^{N_t} Z_i + \sigma B_t$ is RN iff

$$p = r - (\sigma^2/2) - \lambda(\hat{P}(1) - 1)$$

In conclusion, RN pricing of Levy processes for us will amount simply to using only measures satisfying (28).

Surprisingly, pricing binary, asset or nothing options and call options becomes actually easier in this more general framework. We find that:

- <u>**Binary**</u> options with payoff $1_{\{S_T \geq K\}} = 1_{\{X_T \geq a\}}$,where $a = \log(\frac{K}{S_0})$, have the value:

  $$e^{-rT} \Psi_{\theta}(a)$$

  where Ψ_{θ} is the RN Escher transformed distribution of Y_T

- **Asset or nothing** options with payoff $S_T 1_{\{S_T \geq K\}}$ have the value:

$$S_0 \Psi_{\theta+1}(a)$$

(since $e^{-rT}\mathbb{E}S_T e^{\theta X_t - t\,c(\theta)} 1_{\{S_T \geq K\}} = S_0 \mathbb{E} e^{(\theta+1)X_t - t\,c(\theta+1)} 1_{\{S_T \geq K\}}$)

- **Call** options with payoff $(S_T - K)_+$ have the value:

$$S_0 \Psi_{\theta+1}(a) - e^{-rT} K \Psi_\theta(a),$$

which represents a generalized Black Scholes formula for assets modeled by a Levy process.

4.3.5 Path dependent derivatives: Barrier options

Barrier options are options whose payoff depends on whether the underlying asset's price ever reaches a certain level during the contract's period. There are two general categories of barrier options; knock-in-options and knock-out-options. A knock-in barrier option has no value until the underlying price touches a certain barrier and when that happens the option become a plain vanila option. A knock-out barrier option is initially like a plain vanilla option, except that if the price of the underlying asset passes through the stated barrier, the option immediately expires worthless. Some barrier options pay also a rebate when the barrier is reached (instead of expiring worthless), and other barrier options have double barriers. Barrier options can further be categorized by the position of the barrier relative to the initial value of the underlying. If the barrier is above the initial asset price we have an up option; if the barrier is below the initial asset price we have a down option.

Example: Consider an **up and out call** on a stock with initial price S_0 and strike price K and a knock out boundary B. If during the option's life the stock price never rises above B, then the knock out call payoff at expiration is identical to a standard European option, $max\{S_T - K, 0\}$. On the other hand if the stock price does rise above B, then the call is cancelled and the payoff is zero.

Using risk neutral valuation, the value of the European up-and-out call is given by

$$U = \mathbb{E}(e^{-rT} max[0, S_T - K] \;\; I(S_t < B)), \qquad \forall t \in [0, T]$$

where $I(A)$ denotes the indicator function of a set A.

Some other examples are:

- **Down and out binary** with payoff $I(\{L \leq S_t, \forall t \in 0, T\})$
- **Double barrier digital** with payoff $1_{\{L \leq S_t \leq U, \forall t \in 0, T\}}$
- **Down and out Call** with payoff $(S_T - K)_+ 1_{\{L \leq S_t, \forall t \in 0, T\}}$
- **Double barrier call** with payoff $(S_T - K)_+ 1_{\{L \leq S_t \leq U, \forall t \in 0, T\}}$

where L, U are fixed barriers.

Barrier options are becoming increasingly popular because they reduce the cost of plain vanila options while incorporating individual views of the market participants. Moreover, they often have closed form valuation formulas.

Analytical valuation of barrier options requires the availability of formulas for the Q probability of reaching a barrier (for perpetual options) and of the Q distribution at a fixed time of the "absorbed" stock process (for fixed period options).

We note that the problem of valuation of a digital (or binary) barrier option is almost identical with the ruin problem of risk theory the only difference being the stochastic model adopted.

4.3.6 Present value when rates are stochastic: the zero coupon bond

A zero coupon bond is a fixed payment of say, value 1 to be received at a later time T. The present value of a currency unit received T units of time later under the assumption of a constant interest rate r is e^{-rT}.

For a more realistic stochastic model, it is natural to replace the linear term rT by a stochastic process R_T, for example Levy process, under which assumption the value of the zero coupon bond is given by

$$\mathbb{E}e^{-R_T} = e^{-Tc(1)}$$

where $c(u)$ is the cumulant generating function of the Levy process. Assuming $\mathbb{E}R_1 = r$, and an exponential Brownian motion, or an exponential Levy process and taking the first two terms in the Taylor expansion, we get $c(1) \approx r + \frac{\mathrm{Var}\, R_1}{2}$ and thus the present value of the bond

$$e^{-(r+\frac{\mathrm{Var}\, R_1}{2})T}$$

is smaller than in the deterministic case. Variance, which is related to the informal concept of "risk", reduces current values.

Strictly speaking, in the previous subsection on insurance premia it would have been preferable to take into account also discounting in the computation of the premia. This was not done because discounting is not so easy to incorporate in problems related to the reserves process, whose additive nature doesn't mesh well with the multiplicative character of discounting. On the other hand, in the problems concerning the value of assets, which are modeled as multiplicative processes, incorporating a constant discount rate was quite easy.

Modelling the rates themselves as stochastic processes evolving over time is one of the most important current fields of research in mathematical finance.

4.4 Conclusions

In this section we have reviewed the most popular stochastic models used in finance. We have seen how adopting the exponential Brownian motionmodel for assets has lead to one of the most important formulas of Applied mathematics, the Black Scholes formula. This great formula uses only the most elementary stochastic processes theory; in the sense that it only depends on the distribution of the exponential Brownian motionmodel at a fixed time.

In the following chapters we will consider more complicated problems which appear in the valuation of the so called "exotic options", whose valuation requires computing expectations of functionals which depend on the whole path of the stochastic process, like maxima and hitting times

4.5 Exercises

Exercise 4.21

a) Compute the insurance premia provided by the three principles discussed in section 1.1.4. for a general distribution of claims $f(x)$.

b) What do these premia become in the case of exponentially distributed claims with mean β^{-1}?

Exercise 4.22

An insurance company insures a risk with fixed claim sizes M. It is assumed that the total number of claims per year is Poisson distributed with mean 1/2. The premium received annually is M/3. What is the probability that the premium income over six years will be smaller or equal than the payments?

Exercise 4.23 A company with current capital 40000 uses Lundberg's approximation to estimate its probability of being ever ruined. What premium should they charge per year so that the probability of ruin will be no more than $e^{-8} = .00033$, if they estimate $\mathbb{E}X_1 = 100, \operatorname{Var} X_1 = 1000, \lambda = 10$, and use the variance principle as an approximation for the exponential principle?

Exercise 4.24 (Wald's martingale)

Supposing Y_t is a Levy process, find a necessary condition for the process $X_t = e^{uY_t - \delta}$ to be a martingale.

Exercise 4.25 a) Assuming $r = 0$, find the expected payoff (or speculator value) of a "power" option $\mathbb{E}(S_T - K)^2$ for a speculator who believes the stock's distribution is exponential Brownian motion with parameters g, σ.

b) What is the initial risk neutral value of this contract?

c) What is the optimal hedging portfolio for this contract, and what will be the number of stock units and the cash investment in the optimal hedging portfolio at the end of the contract?

Exercise ** 4.26 Compute the expectation, variance and exponential "premium" for the case of a Levy Gamma process whose density $f(x, 1)$ at time 1 is $\Gamma_{(\alpha,\beta)}(x) = \frac{\beta^\alpha x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}$, where $\Gamma(\alpha) = \int_0^i x^{\alpha-1} e^{-\beta x} dx$ is the Gamma funtion.

Exercise ** 4.27

A reinsurance company estimates that an insured aggregate claim process is compound Poisson process with intensity λ and claim sizes Y_i with distribution $\mathbb{P}[Y_i \leq M] = 1 + rlog(1-u)/\lambda, \mathbb{P}[Y_i = M + k] = ru^k/(\lambda k), k = 1, 2....,$ where M is an integer, $r > 0$ and $0 < u < 1$.

The reinsurance company pays out of each claim only the excess over a fixed amount M. Thus, their aggregate claim process is

$$S_t = \sum_{i=1}^{N_t} (Y_i - M)_+.$$

a) Find the moment generating function and the expectation of the aggregate claim process S_t.

b) Assuming an expectation premium principle with loading $\theta = 1$, and $r = 10, u = 1/11$, what is the probability that the total amount of claims is strictly larger than the premium income?

4.6 Solutions

Solution 4.1 a) (b)) We decompose $Y(t + s) = Y(s) + (Y(t + s) - Y(s))$ and taking expectations (variances) we find that both functions satisfy the property $f(t+s) = f(t)+f(s)$ which implies linearity.

Solution 4.2 The independence and stationarity of increments of a Levy process imply that $M(u, t) = \mathbb{E}e^{uY_t}$ satisfies the identity $M(u, t + s) = M(u, t)\ M(u, s)$. Taking logarithms we find that $f(t) = LogM(u, t)$ satisfies the identity $f(t + s) = f(t) + f(s)$ and so $LogM(u, t) = c(u)t$.

Solution 4.3 a) Conditioning on the number of jumps we find

$$\mathbb{E}S_1 = \sum_{j=0}^{\infty} e^{-\lambda} \frac{(\lambda)^j}{j!} j\mathbb{E}X_1 = \mathbb{E}X_1 \mathbb{E}Po(1) = \lambda \int xf(x)dx$$

Similarly, the variance

$$\begin{aligned}
\operatorname{Var} S_1 &= \mathbb{E}(S_1)^2 - (\mathbb{E}S_1)^2 = \\
&= \sum_{j=0}^{\infty} e^{-\lambda} \frac{(\lambda)^j}{j!} \mathbb{E}(\sum_{i=1}^{j} X_i)^2 - (\lambda \mathbb{E}X_1)^2 \\
&= \sum_{j=0}^{\infty} e^{-\lambda} \frac{(\lambda)^j}{j!} (j\mathbb{E}X_1^2 + j(j-1)(\mathbb{E}X_1)^2) - (\lambda \mathbb{E}X_1)^2 \\
&= \lambda \mathbb{E}X_1^2 + \lambda(\mathbb{E}X_1)^2 - \lambda(\mathbb{E}X_1)^2 = \lambda \mathbb{E}X_1^2
\end{aligned}$$

b) Let $M_{X_1}(u) = \mathbb{E}e^{uX_1} = \int_0^\infty e^{ux} f(x)dx$ denote the moment generating function of one jump. Conditioning on the number of jumps we find

$\mathbb{E}e^{uS_t} = \mathbb{E}e^{u(\sum_{t=1}^{N_T} X_t)} = \sum_{j=0}^\infty e^{-\lambda t} \frac{(\lambda t)^j}{j!}(M_{X_1}(u))^j = e^{\lambda t(M_{X_1}(u)-1)} = e^{\lambda t(\int_0^\infty (e^{ux}-1)f(x)dx)}.$

Taking logarithms yields:

$$c(u) = \lambda(M_{X_1}(u) - 1)$$

Solution 4.4 Using the Taylor expansion

$$e^x \approx 1 + x + \frac{x^2}{2!} + ...$$

we find:

$$M_{X_t}(\theta) = \mathbb{E}e^{\theta X_t} = 1 + \theta\mathbb{E}X_t + \frac{\theta^2}{2!}\mathbb{E}X_t^2 + ... = 1 + z.$$

Taking logarithms and using the Taylor expansion

$$ln(1 + x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - ...$$

we find:

$$c(\theta, t) = z - \frac{z^2}{2} = \theta\mathbb{E}X_t + \frac{\theta^2}{2!}\mathbb{E}X_t^2 - \frac{\theta^2(\mathbb{E}X_t)^2}{2} = \theta\mathbb{E}X_t + \frac{\theta^2}{2}\text{Var}\, X_t$$

Solution 4.5

a) $\mathbb{E}e^{uN} = \int_{-\infty}^\infty e^{ux}e^{-\frac{x^2}{2}} \frac{dx}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-\frac{(x-u)^2}{2}} e^{\frac{u^2}{2}} dx = \frac{1}{\sqrt{2\pi}} e^{\frac{u^2}{2}} \sqrt{2\pi} = e^{\frac{u^2}{2}}$

b) A zero mean Gaussian random variable X may be represented as $X = \sigma N$ where $\sigma^2 = \text{Var}\, X$. Thus,

$$\mathbb{E}e^{uX} = \mathbb{E}e^{u\sigma N} = e^{\frac{u^2\sigma^2}{2}} = e^{\frac{u^2 \text{Var}\, X}{2}}$$

c)

$$\mathbb{E}e^{uB_t} = e^{\frac{u^2 t}{2}}$$

Solution 4.6 If $B(t)$ is standard Brownian motion, and $s < t$, then $\mathbb{E}B(s)B(t) = \mathbb{E}B(s)(B(s) + [B(t) - B(s)]) = \mathbb{E}B(s)^2 + \mathbb{E}B(s)\mathbb{E}[B(t) - B(s)] = s + 0 = s.$

Solution 4.7 The moment generating function of Brownian motin with drift is:

$$\mathbb{E}e^{u(gt+\sigma B_t)} = e^{t(gu+u^2\frac{\sigma^2}{2})}$$

and its cumulant generating function is

$$c(u) = gu + u^2\frac{\sigma^2}{2}$$

Solution 4.8 By the decomposition theorem Y_t is a sum of three independent processes $Y_t = pt + \sigma B_t + S_t$ and this implies that the cumulant generating functional is the sum of those of the independent components. Thus,

$$c(u) = pu + \frac{\sigma^2}{2}u^2 + \lambda(M_X(u) - 1)$$

Solution 4.9 a) The cumulant generating function must satisfy

$$c(1) = 0$$

b) $\mu = -\frac{\sigma^2}{2}$

Solution 4.10 a) The cumulant generating function must satisfy

$$c(1) = r$$

b) $\mu = r - \frac{\sigma^2}{2}$.

Solution 4.11

1. $\mathbb{E}S_0 e^{(g\,t+\sigma B(t))} = \mathbb{E}S_0 e^{g\,t}e^{\sigma\,\sqrt{t}N} = S_0 e^{g\,t}e^{\frac{1}{2}\sigma^2 t} = S_0 e^{(g+\frac{1}{2}\sigma^2)t}$
2. $\mathbb{P}\{S_t \leq x\} = \mathbb{P}\{S_0 \exp{(g\,t + \sigma B_t)} \leq x\} = \mathbb{P}\{+\sigma\sqrt{t}N) \leq ln(\frac{x}{S_0}) - g\,t\} = \Phi(\frac{ln(\frac{x}{S_0})-g\,t}{\sigma\sqrt{t}}$
3. The density $p(x,t)$ is proportional to $\frac{e^{-\frac{(ln(x)-gt)^2}{2\sigma^2 t}}}{x}$ (called lognormal density). Letting $E(x)$ denote the exponential (so that $p(x,t) = \frac{E(x)}{x}$) and setting the derivative to 0 we get $E(x)(-\frac{(ln(x)-gt)}{\sigma^2 t}\frac{1}{x}x \quad - 1) = 0$ and we find that the mode is $e^{(g-\sigma^2)t}$.
4. We have to find m so that $\mathbb{P}\{S_t \leq m\} = \mathbb{P}\{e^{gt+\sigma B_t} \leq m\} = \mathbb{P}\{gt + \sigma B_t \leq ln(m)\} = \mathbb{P}\{\sigma B_t \leq ln(m) - gt\} = \frac{1}{2}$. Since σB_t is a symetric normal variable, its median is 0 and the last equation can only be satisfied if $ln(m) - gt = 0$ and $m = e^{gt}$.

Solution 4.12 a) We use here the representation: $S_t = S_0 e^{gT+\sigma B(T)} = S_0 e^{gT+\sigma\sqrt{T}N}$.

The present "speculator" value of the binary option is: $\mathbb{E}e^{-rT}I_{\{S_T\geq K\}} = e^{-rT}\mathbb{E}I_{\{gT+\sigma\sqrt{T}N\geq ln(K/S_0)\}} = e^{-rT}\mathbb{E}I_{\{N\geq\frac{ln(K/S_0)-gT}{\sigma\sqrt{T}}\}} = e^{-rT}\mathbb{P}\{-N \leq \frac{ln(S_0/K)+gT}{\sigma\sqrt{T}}\} = e^{-rT}\Phi(\frac{ln(S_0/K)+gt}{\sigma\sqrt{T}})$ where Φ denotes the standard normal c.d.f.

Solution 4.13 Using again the representation $S_T = S_0 e^{gT+\sigma\sqrt{T}N}$ where N is standard normal, we find the initial value of the asset or nothing option as:

$$e^{-rT}\mathbb{E}S_T 1_{\{S_T\geq K\}} = e^{-rT}\mathbb{E}S_0 e^{gT+\sigma\sqrt{T}N} I_{\{N\geq \frac{ln(K/S_0)-gT}{\sigma\sqrt{T}}\}} = S_0 e^{(g-r)T}\int_z^\infty e^{\sigma\sqrt{T}x}e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}}$$

where $z = \frac{ln(K/S_0)-gT}{\sigma\sqrt{T}}\}$. Completing the square, we get:

$$S_0 e^{(g-r+\frac{\sigma^2}{2})T}\int_z^\infty e^{-\frac{(x-\sigma\sqrt{T})^2}{2}}\frac{dx}{\sqrt{2\pi}} = S_0 e^{(g-r+\sigma^2/2)T}\,\Phi(-(z-\sigma\sqrt{T}) = S_0 e^{(g-r+\sigma^2/2)T}\,\Phi(\frac{\log\frac{S_0}{K}+(g+\sigma^2)T}{\sqrt{\sigma^2 T}}).$$

Solution 4.14 The call payoff is a combination of the asset or nothing and binary payoffs:

$$c(S_0, K, T) = \mathbb{E}e^{-rT}(S_T - K)_+ = \mathbb{E}e^{-rT}S_T I_{S_T\geq K} - K\mathbb{E}e^{-rT}I_{S_T\geq K},$$

each of which has already been computed, so we simply need to add up the results, yielding:

$$S_0 e^{(g-r+\sigma^2/2)T}\,\Phi(\frac{\log\frac{S_0}{K}+(g+\sigma^2)T}{\sqrt{\sigma^2 T}}) - Ke^{-rT}\Phi(\frac{\log\frac{S_0}{K}+gT}{\sqrt{\sigma^2 T}})$$

Solution 4.15

In the case of the risk neutral measure $g = r - \sigma^2/2$ and the formula of the expectation for the call option simplifies to:

$$c(S_0, K) = S_0\Phi(\frac{\log\frac{S_0}{K_T}+\frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}) - K_T\Phi(\frac{\log\frac{S_0}{K_T}-\frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}) = S_0\Phi(L) - K\Phi(l)$$

where we put $K_T = Ke^{-rT}$ and $L, l = \frac{\log\frac{S_0}{K_T}\pm\frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}$. This is the celebrated Black Scholes formula for the pricing of call options when the interest rate is r.

Solution 4.16 The risk neutral probabilities in the binomial model when $r = 0$ are:

$$q_u = \frac{s_0 - s_d}{s_u - s_d}$$
$$q_d = \frac{s_u - s_0}{s_u - s_d}$$

Solution 4.17 The forward

a) $\mathbb{E}e^{-rT}S_T = e^{-rT}\mathbb{E}(S_0 + gT + \sigma B_T) = e^{-rT}(S_0 + gT)$.

b) $\mathbb{E}e^{-rT}S_T = \mathbb{E}S_0 e^{(g-r)T+\sigma B_T} = S_0 e^{(g-r+\frac{\sigma^2}{2})T}$.

c) $\mathbb{E}S_T = \mathbb{E}S_0 e^{-\frac{\sigma^2}{2}T+\sigma B_T} = S_0 e^{(-\frac{\sigma^2}{2}+\frac{\sigma^2}{2})T} = S_0$.

The risk neutral initial value of a forward coincides with the current value of the asset.

d) Similarly, the risk neutral value of a forward at time t is given by S_t. Thus, the partial with respect to S_t is one, and so by the fundamental theorem of derivative pricing the optimal hedging portfolio at time t will contain $\Delta_t = 1$ stock unit. The value of the cash investment $V_t - \Delta_t S_t$ is $S_t - 1S_t = 0$. Thus, the hedging strategy is simply "buy and hold".

Solution 4.18 ** The partial of the Black Scholes formula...

Solution 4.19 ** The Escher transform....

Solution 4.20 Let $\mathbb{E}X_1, \operatorname{Var} X_1, M_{X_1}(\theta)$ denote the expectation, variance and moment generating function of the jump distribution.

Using the results of exercise 1.3, we find that:

1. The expectation principle yields

$$p = (1+\theta)\mathbb{E}S_1 = (1+\theta)m\lambda$$

2. The variance principle yields

$$p = \lambda(\mathbb{E}X_1 + \frac{\theta}{2}\mathbb{E}X_1^2).$$

3. The exponential principle yields

$$p = \frac{\lambda(M_{X_1}(\theta) - 1)}{\theta}$$

In the exponential case $\varphi(x) = \beta e^{-\beta x}$ the formulas simplify further by plugging $\mathbb{E}X_1 = \beta^{-1}, \mathbb{E}X_1 = 2\beta^{-2}, M_{X_1}(\theta) = \frac{\beta}{\beta-\theta}$.

Solution 4.21 The expected number of claims over six years is $6\frac{1}{2} = 3$. The aggregate claim over six years will be MP_3, where P_3 denotes a Poisson r.v. with mean $6\frac{1}{2} = 3$. The premium income will be $6\frac{M}{3} = 2M$. Thus

$$\mathbb{P}[2M \leq S] = \mathbb{P}[2 \leq P_3] = 1 - e^{-3} - 3e^{-3}$$

Solution 4.22 Lundberg's approximation

Lundberg's approximation consists in assuming that if the premium charged is provided by the exponential principle $p = \frac{c(\theta)}{\theta}$ then the probability of ruin is given by:

$$\varphi(u) \approx e^{-\theta u}$$

Hence, θ is determined by $e^{-400000\theta} = e^{-8}$, and $\theta = \frac{1}{5000}$.

The variance approximation of the exponential principle is $\lambda(\mathbb{E}X_1 + \frac{\theta}{2}\text{Var}\, X_1) = 10(100 + \frac{1000}{10000}) = 10001$.

Solution 4.23 The martingale condition implies constancy of the expectations $\mathbb{E}e^{-\delta t + uY_t} = e^{-\delta t + tc(u)} = 1$ which yields

$$c(u) = \delta$$

Solution 4.24 We will use here the formula of the moment generating function of a Brownian motion $\mathbb{E}e^{aB_{g,\sigma^2}(t)} = e^{t(ga + \frac{\sigma^2 a^2}{2})}$ (which may be established by completing the square or by writing $B_{g,\sigma^2}(t) = gt + \sigma\sqrt{t}N$).

$$\begin{aligned}\mathbb{E}(S_T - K)^2 &= K^2 - 2K\mathbb{E}[S_0 e^{B_{g,\sigma}(T)}] + \mathbb{E}[S_0^2 e^{2\,B_{g,\sigma}(T)}] \\ &= K^2 - 2KS_0 e^{(g+\frac{\sigma^2}{2})T} + S_0^2 e^{(2g+2\sigma^2)T}\end{aligned}$$

b) In the case of martingale geometric Brownian motion with $g = -\sigma^2/2$ (the case mainly used in derivatives pricing) this result becomes:

$v_0 = \mathbb{E}(S_T - K)^2 = K^2 - 2KS_0 + S_0^2 e^{\sigma^2 t}$.

c) THe value at time t of the hedging portfolio should be

$V_t = \mathbb{E}[(S_T - K)^2/S_t] = K^2 - 2KS_t + S_t^2 e^{\sigma^2(T-t)}$.

The number of stock units held and the cash investment are respectively: $2S_t e^{\sigma^2(T-t)} - 2K, K^2 - S_t^2 e^{\sigma^2(T-t)}$, and the expiration values are: $2(S_T - K), K^2 - S_T^2$.

5 The method of difference equations for computing expectations

5.1 Difference (recurrence) equations for expectations of simple random walks

In this section we will learn how to compute various functionals of Markov chains (more specifically random walks). Among others, we will visit a famous gambling problem called the **The Gambler's ruin**. In this problem we are interested in a) the probability of winning K (say a million pounds) before getting bankrupt, and b) the expected time until we either win or become bankrupt. The idea of the method is to derive recurrence equations which relate the value of these functions when starting from different neighboring points.

Example 5.1 (The Gambler's ruin probability) The position of a **simple random walk** at time t is given by $X(t) = \sum_{i=1}^{t} Z_i$ where each step may be ± 1, with $P[Z = 1] = p$ and $P[Z = -1] = q = 1-p$. Let T be the exit time from $[0, K]$, i.e. $T = min(T_0, T_K)$, where T_0 , T_K are the first hitting times of 0 and K, respectively. The gambler's ruin problem is to find $p_n = P_n[X(T) = K]$, i.e. the probability of not ending ruined starting at $X(0) = n$.

By conditioning, we find that p_n must satisfy

$$\begin{aligned} p_n &= p\, p_{n+1} + q\, p_{n-1} \quad 1 \leq n \leq K-1 \\ p_0 &= 0 \\ p_K &= 1 \end{aligned}$$

The method to solve this type of problems called recurrence equations with constant coefficients is described in more detail in the Appendix. It involves looking for solutions of the form $p_n = r^n$. If $p \neq q$ we find two distinct roots $r_1 = q/p, r_2 = 1$. The general solution is then $p_n = k_1 r_1^n + k_2 r_2^n = k_1 (q/p)^n + k_2$ where k_1, k_2 are detremined from the boundary conditions. The final solution (called harmonic function with given boundary conditions) is $p_n = \frac{(q/p)^n - 1}{(q/p)^K - 1}$. When $p = q = 1/2$ we find that the solution is linear $p_n = \frac{n}{K}$.

This method is important because it is very rarely possible to compute expectations and probabilities in explicit form and the best we can usually get is equations (nonlinear, or differential) they satisfy. Luckily, in problems about Markov processes it is always possible to obtain **difference** or **differential equations** which may then be easily solved numerically by the use of computers.

The method to find the recurrence equations, **conditioning on the result of the first step**, is illustrated below in some simple problems which can also be solved explicitly (though again, the virtue of the method is that it works also when explicit answers are unavailable). In the first example below we will revisit the gambler's ruin problem.

Example 5.2 Ruin probability on unbounded state space We consider now a gambler whose winning probability p is greater than q and who wants to gamble forever, unless he is "ruined". Let φ_n denote the ruin probaility, i.e $\varphi_n = \mathbb{P}_n\{X_t = 0 \text{ for some } t \geq 0\}$. This problem may be solved by taking the complememntary of the result in Example 1.1,

$1-p_n$ and letting K go to ∞. A simpler solution is to add an appropriate boundary condition at ∞, as below.

$$\begin{aligned} \varphi_n &= p\varphi_{n+1} + q\varphi_{n-1} \\ \varphi_0 &= 1 \\ \varphi_\infty &= 0 \end{aligned}$$

Note: The boundary condition at ∞ follows intuitively from the fact that $p > q$. In the opposite case $p < q$ we would put $\varphi_\infty = 1$ (by the law of large numbers).

The general solution is the same as in example 1, $\varphi_n = k_1(q/p)^n + k_2$; the second boundary condition yields $k_2 = 0$ and the first yields $k_1 = 1$. The final solution is $\varphi_n = \frac{(q/p)^n}{.}$

Note: The fact that $\varphi_n = \varphi_1^n$ has a clear probabilistic intrepretation. Getting ruined with an initial capital of n involves "slipping" down by one n times, and these "slips" are independent and have all the same probability of φ_1.

Example 5.3 The expected value of the geometric random variable with a "deadline"

Let T denote the number of heads preceding the first tail in the tossing of a coin. The method of conditioning yields an equation for $\varphi = \mathbb{E}T$.

$$\varphi = p(1 + \varphi) + q0$$

which yields the well known result: $\varphi = \frac{p}{q}$.

Let now $\tilde{T} = \min(T, N)$ denote the geometric truncated at a fixed later time N. We want to compute again the expected number of heads before tails and before the fixed "deadline".

Before we apply conditioning, it is important to realize that this time the functional depends on the remaining number of steps n until the deadline. We must solve

$$\begin{aligned} \varphi_n - p\varphi_{n-1} &= p \\ \varphi_0 &= 0. \end{aligned}$$

Following the general method for nonhomogeneous equations described in the Appendix, we find first the solution of the homogeneous equation which is Ap^n; then we look for a particular constant solution (since the RHS is constant), getting $\frac{p}{q}$. Thus, the general solution of this equation is:

$$\varphi_n = Ap^n + \frac{p}{q}.$$

The boundary condition yields $A = -\frac{p}{q}$. Note the case $n = \infty$ yields the no deadline result $\frac{p}{q}$ and $n = 1$ yields $\varphi_1 = p$, which may be checked directly.

Example 5.4 (The expected gambling time) The expected total number of steps until the game stops $t_n = E_n T$ (starting at $X(0) = n$) may also be found by conditioning. The corresponding equations

$$\begin{aligned} t_n &= 1 + p\, t_{n+1} + q\, t_{n-1} \quad 1 \le n \le K-1 \\ t_0 &= 0 \\ t_K &= 0 \end{aligned}$$

involve this time the "nonhomogeneous" term 1 (the name relates to the presence of terms which do not involve the unknown sequence t_n).

To solve this nonhomogeneous problem, we recall first from the previous example that if $p \neq q$ the general homogeneous solution is $k_1 (q/p)^n + k_2$. Based on the constant nonhomogeneous term 1, our initial particular solution guess would be some constant k, but since that is known to satisfy the homogeneous equation we modify the guess to kn.

Plugging kn in the recurrence equation we find that $0 = 1 + pk - qk$ and so $k = \frac{1}{q-p}$.

We determine next k_1, k_2 by requiring $\frac{n}{q-p} + k_1 (q/p)^n + k_2$ to verify the boundary conditions. We find that

$$t_n = \frac{n}{q-p} - \frac{K}{q-p} \frac{(q/p)^n - 1}{(q/p)^K - 1}.$$

If $p = q = 1/2$ then the general homogeneous solution is $k_1 n + k_2$. To find the particular solution we need now to modify twice the initial guess k (by multiplying by n). The trial particular solution is kn^2.

Plugging in the equation yields $0 = 1 + .5k(2n+1) + .5k(-2n+1) = 1 + k$ and so we find that kn^2 is a particular solution provided that $k = -1$. Finally, the boundary conditions yield $t_n = n(K - n)$.

Note that both questions above involved the same homogeneous part, namely the second order difference operator

$$(Gf)_n = p\, f_{n+1} + q\, f_{n-1} - f_n. \tag{29}$$

In this notation, the system in Example 2.1 is

$$\begin{aligned} (Gp)_n &= 0 \\ p_0 &= 0 \\ p_K &= 1 \end{aligned}$$

and that in Example 2.4 is

$$\begin{aligned} (Gt)_n + 1 &= 0 \\ t_0 &= 0 \\ t_K &= 0 \end{aligned}$$

the only difference between Example 2.1 and Example 2.4 being in the boundary conditions and in the presence of the nonhomogeneous part. This situation is general for the whole theory of expectations of functionals of Markov processes. Each process (each probabilistic specification) has an associated difference operator in the case of discrete processes and associated differential operator in the case of continuous processes; the associated operator appears in all the problems about that process.

Summary: We may associate to the simple random walk X_t an "operator" G defined by (33) (this is related to the transition matrix P via $G = P - I$) so that various types (see below) of problems concerning expectations of X_t reduce to solving difference equations involving this operator. The difference between the various problems is seen in their boundary conditions and nonhomogeneous terms. Some examples are:

(A) Expected final prize functionals $f(x) = \mathbb{E}_x h(X_T)$, where T is the hitting time of some set.

Example 1.1 is of this type, with $h(K) = 1, h(0) = 0$.

These satisfy the homogeneous equation: $Gf = 0$ with boundary condition $f = h$ on the boundary.

(B) Expected total cost functionals $f(x) = \mathbb{E}_x \sum_{t=1}^{T} c(X_t)$, where T is the the hitting time of some set.

Example 1.2 is of this type, with $c(x) = 1$.

These satisfy the inhomogeneous harmonic equation $Gf + c(x) = 0$ with $f = 0$ on the boundary.

(C) The stationary probabilities π satisfy the equation $\pi G = 0$, with π being a vector of probabilities.

Note: The key idea by which the above equations (and others) have been obtained is to identify the set of all possible starting points n ($0 \le n \le K$), to fix a starting point n, and then by conditioning on what may happen after one step to obtain an equation relating the functional of the initial starting point to the functional starting from other neighboring points.

We will consider in the sequel these three types of problems and some other types, under different probabilistic models, to illustrate the theme that each model has a "characteristic" operator G, which appears in all the questions about that model, and that each type of problem has "characteristic" nonhomogeneous terms and boundary conditions.

In the end we will be able to obtain the difference/differential equation for a problem just by "pairing" the operator of the model with the nonhomogeneous term and boundary conditions of the problem.

5.2 Exercises

Exercise 5.1 (Symmetric Random Walk) A particle performs a symmetric random walk $X(t) = \sum_{i=1}^{t} Z_i$ ($P[Z = 1] = P[Z = -1] = 1/2$) on the integers between 0 and K, starting at $X(0) = x$. Let T be the exit time from $[0, K]$, i.e. $T = min(T_0, T_K)$, where T_0 , T_K are the number of steps until the first hitting times of 0 and K, respectively. Using the method of conditioning on the position after one step, it is easy to check that the various functionals of the starting point below must satisfy the difference equations provided. Solve the respective equations.

[(a)] $p_x = P_x[X(T) = K]$ (Hitting distribution) satisfies:

$$\begin{aligned} p_x &= \frac{p_{x+1}}{2} + \frac{p_{x-1}}{2} \quad \text{for any } 1 \leq x \leq K-1 \\ p_K &= 1 \\ p_0 &= 0 \end{aligned}$$

[(b)] $w_x = E_x[X(T)]$ (expected final value) satisfies:

$$\begin{aligned} w_x &= \frac{w_{x+1}}{2} + \frac{w_{x-1}}{2} \quad \text{for any } 1 \leq x \leq K-1 \\ w_K &= K \\ w_0 &= 0 \end{aligned}$$

[(c)] $t_x = E_x[T]$ (expected hitting time) satisfies:

$$\begin{aligned} t_x &= \frac{t_{x+1}}{2} + \frac{t_{x-1}}{2} + 1 \quad \text{for any } 1 \leq x \leq K-1 \\ t_K &= 0 \\ t_0 &= 0 \end{aligned}$$

[(d)] $i_x = E_x[\sum_0^T X(t)]$ (expected total inventory cost) satisfies:

$$\begin{aligned} i_x &= \frac{i_{x+1}}{2} + \frac{i_{x-1}}{2} + x \quad \text{for any } 1 \leq x \leq K-1 \\ i_K &= 0 \\ i_0 &= 0 \end{aligned}$$

[(e)] Consider the biased random walk on the numbers between 0 and K with the "reflection end rules" $P_0[X(1) = 0] = a$, and $P_K[X(1) = K] = a$. Conditioning on the position of the walk one step before, we find that the stationary distribution of the walk π_x satisfies the equilibrium equations $\pi_x = \sum_y \pi_y P_{y,x}$ which in this case are:

$$\begin{aligned}
\pi_x &= \frac{\pi_{x+1}}{2} + \frac{\pi_{x-1}}{2} \quad \text{for any } 2 \le x \le K-2 \\
\pi_1 &= (1-a)\pi_0 + \frac{\pi_2}{2} \\
\pi_0 &= \frac{\pi_1}{2} + a\,\pi_0 \\
\pi_{K-1} &= (1-a)\pi_K + \frac{\pi_{K-2}}{2} \\
\pi_K &= \frac{\pi_{K-1}}{2} + a\,\pi_K
\end{aligned}$$

Find the stationary distribution, up to the normalization constant.

Exercise 5.2 "Lazy" Random Walk) Repeat Exercise 1 a-d)if $P[Z = 1] = P[Z = -1] = p$, and $P[Z = 0] = 1 - 2p$ where p is some number less than 1/2.

Exercise 5.3 ("Biased" Random Walk) a) What are the systems of equations investigated in Exercise 1 (a)-(d) if the random walk is asymmetric, with $P_i[Z = 1] = p$, $P_i[Z = -1] = q = 1 - p$ at every interior point i (do not solve the equations).

b) Let $p_{x,K}$ denote the probability that the random walk starting at x will ever reach a positive integer K and $q_{x,K}$ denote the probability of never reaching K (that is of getting lost at $-\infty$). Note that $q_{x,x+1}$, the probability of never visiting the points to the right, is independent of x, and let r denote $p_{x,x+1}$. Argue that $p_{x,K} = r^K$ and that $p_{0,1} = p + qp_{-1,1}$ and thus that r has to satisfy $r = p + q\,r^2$. What is r when $p < q$ and when $p > q$. (To resolve the case $p = q$, more work is necessary; one way is to find $p_{0,1}$ in the presence of an extra lower barrier at L and than let $L \to -\infty$.)

* c) Find $E_0[T_K]$ in the case $p > q$. Hint: You may either add an extra lower barrier at L and solve $t(x) = pt(x+1) + qt(x-1) + 1, t(K) = 0, t(L) = 0$ and than let $L \to -\infty$, or more directly, solve $t(x) = pt(x+1) + qt(x-1) + 1, t(K) = 0, t(-\infty)$ "does not blow up exponentially".

Exercise 5.4 Consider a simple random walk on the cube $[0,1]^3$ starting from the origin 0. Find:

a) $E_u[T_0]$, where $u = (1,1,1)$. (Hint: Set up a system of equations for $E_x[T_0]$ where x can be any starting point, and use symmetry to reduce the number of unknowns to three.)

b) $E_0[\bar{T}_0]$, where $\bar{T}_0$ is the expected time until revisiting 0, starting from 0.

c) $P_a[X(T) = u]$, where $a = (0,0,1)$ and $T = min[T_0, T_u]$

d) The probability starting at 0 that the walk visits $(1,1,1)$ exactly k times before returning at 0. Hint: Consider first $k = 0$, $k = 1, ...$ Check that the sum of the probabilities adds up to 1.

5.3 Apendix: One dimensional linear recurrence equations with constant coefficients

The two second order linear recurrence equations equations below

$$au_{n+2} + bu_{n+1} + cu_n = 0, \tag{30}$$

$$av_{n+2} + bv_{n+1} + cv_n = d_n, \tag{31}$$

are called **homogeneous** and **nonhomogeneous** respectively.

The homogeneous equation

If the coefficients a, b and c are constants, it is known that some of the solutions will be of the form $u_n = x^n$ for all n (exponential functions). To find x we plug x^n in (1) and find that x has to satisfy the **auxiliary equation**:

$$ax^2 + bx + c = 0. \tag{32}$$

Let x_1 and x_2 be the two roots of the quadratic equation (32). It turns out that the *general* solution of (1) is always of the form

1. If $x_1 \neq x_2$
$$u_n = Ax_1^n + Bx_2^n,$$
2. If $x_1 = x_2$,
$$u_n = Ax_1^n + Bnx_1^n,$$
   for some constants A and B.

In either case A and B must be determined from additional boundary conditions.

The nonhomogeneous equation

The solution of the nonhomogeneous problem (2) involves four steps:

1. Find the general solution for the auxiliary homogeneous equation (1).
2. Determine a "trial" form w_n for a particular solution of (2), of the same general form as the right hand side d_n, but with undetermined coefficients. For example, if d_n is a polynomial of order k, try a general polynomial of order k. However, if any of the terms in your trial form matches one in the general solution of the homogeneous equation obtained in Step 1, you must multiply your trial form by n until there is no match.
3. Find the values of the coefficients in w_n by the method of undetermined coefficients.

4. The general solution of (2) is the sum

$$v_n = w_n + u_n$$

Find the as yet undetermined coeficients (in u_n) by applying the boundary conditions for v_n.

5.4 Solutions

Solution 5.1 Symmetric Random Walk

(a) $p_x = P_x[X(T) = K]$ (Hitting distribution) satisfies the "harmonic" system:

$$\begin{aligned} p_x &= \frac{p_{x+1}}{2} + \frac{p_{x-1}}{2} \quad \text{for any } 1 \le x \le K-1 \\ p_K &= 1 \\ p_0 &= 0 \end{aligned}$$

Looking for geometric solutions r^xleads to the equation $r^2 - 2r + 1 = 0$ with two identical roots $r_{1,2} = 1$. The general solution is thus $p_x = A + Bx$. Using the boundary conditions we get $p_x = \frac{x}{K}$.

(b) $w_x = E_x[X(T)]$ (expected final prize) is also an "harmonic function, with different boundary conditions however:

$$\begin{aligned} w_x &= \frac{w_{x+1}}{2} + \frac{w_{x-1}}{2} \quad \text{for any } 1 \le x \le K-1 \\ w_K &= K \\ w_0 &= 0 \end{aligned}$$

The solution $w_x = x$ may be obtained as above, or using the optional stopping theorem and the fact that $X(t)$ is a martingale.

(c) $t_x = E_x[T]$ (expected hitting time) satisfies the inhomogeneous system

$$\begin{aligned} t_x &= \frac{t_{x+1}}{2} + \frac{t_{x-1}}{2} + 1 \quad \text{for any } 1 \le x \le K-1 \\ t_K &= 0 \\ t_0 &= 0 \end{aligned}$$

The general homogeneous solution (harmonic function) $A + Bx$ for this operator has already been obtained above. We look then for particular solutions of the same form as the R.H.S, i.e. $t(x) = C$; however because constants (and linear functions) are harmonic, we have to modify twice our guess, ending up with the trial solution $t(x) = Cx^2$. Plugging this in the system we find the particular solution $t(x) = -x^2$.

The general solution is $t(x) = -x^2 + A + Bx$ and after using the boundary conditions we find $t_x = x(K - x)$.

(d) $i_x = E_x[\sum_0^T X(t)]$ (expected total inventory cost) satisfies the inhomogeneous system:

$$\begin{aligned} i_x &= \frac{i_{x+1}}{2} + \frac{i_{x-1}}{2} + x \quad \text{for any } 1 \le x \le K-1 \\ i_K &= 0 \\ i_0 &= 0 \end{aligned}$$

The homogeneous solution is still the same. The particular non homogeneous solution is found by the method of undetermined coefficients to be $\frac{-x^3}{3}$. Finally, the boundary conditions yield $i(x) = \frac{x(K^2-x^2)}{3}$.

Remark: The questions (a-d) above involve all the same second order difference operator $(Gf)_x = \frac{f_{x+1}}{2} + \frac{f_{x-1}}{2} - f_x$.

In this notation, (a), (b) are $(Gf)_x = 0$, (c) is $(Gf)_x + 1 = 0$ and (d) is $(Gf)_x + x = 0$. We will look below at the same questions under different probabilistic assumptions, and the conclusion is that each model has its own operator G, which appears in all the questions about that model.

(e) Note first that for this question we have to provide information on the behavior of the particle after hitting the boundary, since for determining the stationary distribution we have to assume that the walk goes on forever, whereas in the previous questions it stopped upon hitting the boundary. We chose a symmetric situation $P_0[X(1) = 0] = a$ and $P_K[X(1) = K] = a$, a representing the probability of "resting" at the boundary. We will see that the answer does depend on a.

The equilibrium equations for the stationary distribution of the walk π_x may be obtained by conditioning on the position of the particle one step before:

$$\begin{aligned}
\pi_x &= \frac{\pi_{x+1}}{2} + \frac{\pi_{x-1}}{2} \quad \text{for any } 2 \le x \le K-2 \\
\pi_1 &= (1-a)\pi_0 + \frac{\pi_2}{2} \\
\pi_0 &= \frac{\pi_1}{2} + a\,\pi_0 \\
\pi_{K-1} &= (1-a)\pi_K + \frac{\pi_{K-2}}{2} \\
\pi_K &= \frac{\pi_{K-1}}{2} + a\,\pi_K
\end{aligned}$$

We try to express all the probabilities as functions of p_0 (which may be determined by the normalization condition). We guess that by symmetry all probabilities should be equal, except the boundary ones. The third equation yields $\pi_1 = 2(1-a)\pi_0$. Upon plugging the formula $\pi_1 = 2(1-a)\pi_0$ in the second equation we find that $\pi_2 = 2(1-a)\pi_0 = \pi_1$. The first equation shows that π_x has to be linear, for any $1 \le \text{x} \le \text{K} - 1$. Thus, $\pi_1 = \pi_2 = \cdots = \pi_{K-1} = 2(1-a)\pi_0$ and $\pi_K = \pi_0$ from the last equation.

Note that we need not try to use the equation at $x = K-1$, (except as a check) since we know that the equilibrium equations have always one redundant equation (which should be replaced by the normalization condition).

Finally, the normalization condition yields: $\pi_0 = \pi_K = \frac{1}{2(K+a(1-K))}$. Note that the total mass at the ends $\pi_0 + \pi_K$ depends on a, being one for $a = 1$ and $\frac{1}{K}$ for $a = 0$. In the symmetric case $a = 1/2$ all the probabilities equal $\frac{1}{K+1}$

Solution 5.2 "Lazy" Random Walk

(a,b) Both p_x and w_x do not change, since the systems they satisfy can be manipulated to the previous form. p_x for example satisfies:

$$\begin{aligned}
2pp_x &= pp_{x+1} + pp_{x-1} \quad \text{for any } 1 \le x \le K-1 \\
p_K &= 1 \\
p_0 &= 0
\end{aligned}$$

(c) $t_x = E_x[T]$ (expected hitting time) satisfies

$$\begin{aligned} t_x &= \frac{t_{x+1}}{2} + \frac{t_{x-1}}{2} + \frac{1}{2p} \quad \text{for any } 1 \le x \le K-1 \\ t_K &= 0 \\ t_0 &= 0 \end{aligned}$$

whose solution is $t_x = \frac{x(K-x)}{2p}$.

(d) $i_x = E_x[\sum_0^T X(t)]$ (expected total inventory cost) satisfies

$$\begin{aligned} i_x &= \frac{i_{x+1}}{2} + \frac{i_{x-1}}{2} + \frac{x}{2p} \quad \text{for any } 1 \le x \le K-1 \\ i_K &= 0 \\ i_0 &= 0 \end{aligned}$$

whose solution is: $i(x) = \frac{x(K^2-x^2)}{6p}$.

Note: The operator of lazy random walk is $(G_L f)_x = (2p)(\frac{f_{x+1}}{2} + \frac{f_{x-1}}{2} - f_x) = (2p)(Gf)_x$, where G is the operator for the symmetric random walk.

Solution 5.3 "Biased" Random Walk

Let us put $(Gp)_x = p\, p_{x+1} + q\, p_{x-1}$

(a) The respective equations are:

1. $p_x = P_x[X(T) = K]$ (Hitting distribution) satisfies:

$$\begin{aligned} p_x &= (Gp)_x \quad \text{for any } 1 \le x \le K-1 \\ p_K &= 1 \\ p_0 &= 0 \end{aligned}$$

2. $w_x = E_x[X(T)]$ (expected final value) satisfies

$$\begin{aligned} w_x &= (Gw)_x \quad \text{for any } 1 \le x \le K-1 \\ w_K &= K \\ w_0 &= 0 \end{aligned}$$

3. $t_x = E_x[T]$ (expected hitting time) satisfies

$$\begin{aligned} t_x &= (Gt)_x + 1 \quad \text{for any } 1 \le x \le K-1 \\ t_K &= 0 \\ t_0 &= 0 \end{aligned}$$

4. $i_x = E_x[\sum_0^T X(t)]$ (expected total inventory cost) satisfies

$$\begin{aligned} i_x &= (Gi)_x + x \quad \text{for any } 1 \le x \le K-1 \\ i_K &= 0 \\ i_0 &= 0 \end{aligned}$$

(b) Let $p_{x,K}$ denote the probability that the random walk starting at x will ever reach a positive integer K and $q_{x,K}$ denote the probability of never reaching K (that is of getting lost at $-\infty$). Now $q_{x,x+1}$ (and hence $px, x+1$ also) is independent of x, since the probability of never leaving a half line cannot depend on the label we put to its right end. Letting r denote $p_{x,x+1}$, we see by the Markov property that $p_{x,K} = r^{K-x}$.

Since $p_{0,1} = p + qp_{-1,1}$ and thus $r = p + q\, r^2$, we find that r may equal either $\frac{p}{q}$ or 1. By the law of large numbers, the first case will occur if $p < q$ and the second when $p > q$. (To resolve the case $p = q$, more work is necessary; one way is to find $p_{0,1}$ in the presence of an extra lower barrier at L and then let $L \to -\infty$.)

(c) We have to solve:

$$t(x) = p\, t(x+1) + q\, t(x-1) + 1$$
$$t(-\infty) \text{ "does not blow up exponentially"}$$
$$t(K) = 0$$

The roots of the auxiliary equation are $\frac{q}{p}$ and 1. The particular solution is $\frac{-x}{p-q}$ and the general solution of the nonhomogeneous recurrence is is $t(x) = a(q/p)^x + b + \frac{-x}{p-q}$. The first term blows up exponentially at $-\infty$ and so $a = 0$. Using the boundary condition we get $t(x) = \frac{K-x}{p-q}$. Thus $t(0) = \frac{K}{p-q}$.

5.4 $X(t)$ denotes simple random walk on the cube $[0,1]^3$, $\vec{0}$ denotes the origin $(0,0,0)$, the opposite corner $(1,1,1)$ is denoted by u. We note a symmetric role in all the questions for all the neighbours of the origin, which allows us to denote them by the same letter $a = (0,0,1), \ldots$. Similarly, we denote the neighbours of u by $b = (0,1,1), \ldots$.

a) To find $t_u = E_u[T_0]$, we solve the system:

$$t_u = 1 + t_b$$
$$t_b = 1 + \frac{2}{3}t_a + \frac{1}{3}t_u$$
$$t_a = 1 + \frac{2}{3}t_b$$

whose solution is $t_a = 7, t_b = 9, t_u = 10$.

b) $E_0[\bar{T}_0]$ where $\bar{T}_0$ is the expected time until revisiting 0 starting from 0, is given by $1 + t_a = 1 + 7 = 8$. Note that is precisely the inverse of the long run probability of being at 0, which is a well known result on expected return times.

c) $P_a[X(T) = u]$, is obtained from the solution of

$$p_a = \frac{2}{3}p_b$$
$$p_b = \frac{2}{3}p_a + \frac{1}{3}$$

which is $p_b = \frac{3}{5}, p_a = \frac{2}{5}$.

d) Let p_k be the probability of exactly k visits to $(1,1,1)$ before returning to 0. Then p_0 is the same as the probability starting at a that the walk returns to 0 before visiting $(1,1,1)$, which is $\frac{3}{5}$.

$p_1 = (\frac{2}{5})^2$, $p_2 = (\frac{2}{5})^2(\frac{3}{5})$, and in general $p_k = (\frac{2}{5})^2(\frac{3}{5})^{k-1}$. (Check that $\sum_{k=1}^{\infty} p_k = \frac{2}{5}$, as it should).

6 Differential equations for functionals of Brownian motion

In some ways, Brownian motion is a very complicated process; it has very "wiggly" paths, with quite unusual properties. For example, its path is nowhere differentiable, and after reaching any level, its graph will cross that level an infinite number of times in any neighborhood, no matter how small, of the first crossing.

Despite these "path" complexities, problems of computing expectations or probabilities may always be reduced to the relatively easy task of solving differential equations (ordinary or partial). †

These differential equations maybe obtained by passing to the limit in the random walks approximations for standard Brownian motion and Brownian motion with drift, which were introduced in Sections 1.1.5, 1.1.6 respectively. ‡

Definition: Infinitesimal random walk is a process time intervals of h. We found out in Section 1.1.5 that symmetric random walk may converge to a finite limit iff $D^2 = \sigma^2 h$, for some σ. This type of scaling needs to be enforced because it ensures that the variance of the process $S_h(t)$ converges to a finite limit when $h \to 0$.

(Note that $\operatorname{Var} S_h(t) \approx \frac{t}{h}\mathbb{E}Z_1^2 = t\frac{D^2}{h}$.)

To approximate Brownian motion without drift σB_t (where B_t is standard Brownian motion which corresponds to the subcase $\sigma = 1$.) we employ **symmetric infinitesimal random walk** with equal probabilities for going up(down), and to approximate Brownian motion with drift $\mu t + \sigma B_t$ we employ **infinitesimal random walk with vanishing bias** with probabilities for going up(down) of $p, q = \frac{1}{2} \pm \frac{\mu}{2\sigma^2} D$ (which converge both to $\frac{1}{2}$ when $D \to 0$).

6.1 Ito's formula for Brownian motion

We will establish first Ito's formula for symmetric infinitesimal random walk and then pass to the limit.

To motivate the result, consider first the problem of determining the probability $p(x) = \mathbb{P}_x\{T_K \le T_L\}$ of exiting through the upper barrier for symmetric infinitesimal random walk.

†The fact that we may practically bypass all the complexities of Brownian motion is quite reassuring, if we bear in mind that Brownian motion doesn't actually exist in real life. In practice, we always only observe random walk and model it as Brownian motion or as Levy process with jumps; clearly, discussing whether the random walk we observe is indeed continuous (i.e Brownian motion) or rather whether it jumps during the periods of time when we can't observe it is rather an academic topic.

‡In fact, the property that computing expectations maybe reduced to solving various types of differential problems is shared by all stationary Markov processes, which may all be characterized by a differential, integral or difference operator G called **generator**. Our results in this section are purposely formulated in terms of this operator, so that the result for other processes may be obtained simply by substituting the corresponding generator.

Just as in the case $D = h = 1$, we see that $p(x)$ must satisfy

$$p(x) = 1/2p(x+D) + 1/2p(x-D)$$
$$p(K) = 1$$
$$p(L) = 0$$

We could solve this as a difference equation just as in the case $D = h = 1$, but we want to make here the point that things get slightly easier when $D \to 0$, due to the following

Exercise 6.1 Ito's lemma for infinitesimal random walk

Show that for any function $f(S_h(t)$ applied to a process $S_h(t)$ which jumps right or left by $dx = D$ after time intervals of size $dt = h$, where $D, h \to 0$ such that $D^2 = \sigma^2 h$ we have:

a) $1/2f(x+D) + 1/2f(x-D) - f(x) \approx \frac{D^2}{2} f''(x) \approx hGf(x)$

where $Gf(x) = \frac{\sigma^2 f''(x)}{2}$.

b) The expression approximated above is precisely the expected conditional differential $\mathbb{E}[df(S_h(t)/S_h(t) = x] = \mathbb{E}[(f(S_h(t+h)) - f(S_h(t)))/S_h(t) = x]$ of the function $f(S_h(t))$ over a small interval of size h, i.e:

$$\mathbb{E}[df(S_h(t)/S_h(t) = x] = 1/2f(x+D) + 1/2f(x-D) - f(x)$$

c) $$\mathbb{E}[d\ f(S_h(t)/S_h(t) = x] \approx hGf(x)$$

d) The expected new value of the function $f(S_h(t))$ after a small interval of size h is: $\mathbb{E}[f(S_h(t+h)/S_h(t) = x] = f(x) + hGf(x)$

Solution 6.1 a) Using the Taylor expansions: $f(x+D) \approx f(x) + f'(x)D + \frac{f''(x)}{2} D^2$, $f(x-D) \approx f(x) - f'(x)D + \frac{f''(x)}{2} D^2$, we find that for D small we have

$$\frac{f(x+D)}{2} + \frac{f(x-D)}{2} - f(x) \approx \frac{D^2}{2} f''(x)$$

Using the scaling, $D^2 = \sigma^2 h$ we obtain finally "Ito's lemma":

$$\frac{f(x+D)}{2} + \frac{f(x-D)}{2} - f(x) \approx hGf(x)$$

b) $\mathbb{E}[df(S_h(t)/S_h(t) = x] = \mathbb{E}[(f(S_h(t+h)) - f(S_h(t)))/S_h(t) = x] = \frac{f(x+D)}{2} + \frac{f(x-D)}{2} - f(x)$

c) Put a), b) together.

d) This follows from $f(S_h(t+h)) = f(S_h(t)) + f(S_h(t+h)) - f(S_h(t)) = f(S_h(t)) + df(S_h(t))$ by taking expectations and using c):

$$\mathbb{E}[f(S_h(t+h))/S_h(t)=x] = \mathbb{E}[f(S_h(t)) + df(S_h(t))/S_h(t)=x] \approx f(x) + hGf(x)$$

In the limit $h \to 0$, symmetric infinitesimal random walk converges to Brownian motion with variability σ and Exercise 1 yields:

Lemma 6.1. Ito's formula for Brownian motion with variability σ

Let $f(x)$ denote an arbitrary twice differentiable function applied to Brownian motion with variability σ, i.e. $X_t = \sigma B_t$. Then, the expected value of the function after a small time interval may be approximated by:

$$\mathbb{E}[f(X_{t+h})/X_t = x] \approx f(x) + hGf(x)$$

where $Gf(x) = \frac{\sigma^2}{2} f''(x)$.

It is interesting to compare this result with the classical calculus "approximation" of the differential of a function applied to a deterministic linear process $X_t = \mu t$. In that case we have:

Lemma 6.2. First order Taylor approximation for $df(X_t)$ when X_t is linear motion $X_t = x_0 + \mu t$

Let $f(x)$ denote an arbitrary differentiable function applied to linear motion $X_t = \mu t$. Then, given that $X_t = x$, the value of the function after a small time interval may be approximated by:

$$f(X_{t+h}) = f(x + \mu h) \approx f(x) + h\mu f'(x) = f(x) + hGf(x)$$

where $Gf(x) = \mu f'(x)$.

We take now a leap of faith to the case of Brownian motion with drift $X_t = \mu t + \sigma B_t$, which is composed from a linear drift μt and a Brownian motion with variability σ. It turns out that the generator is precisely the sum of the two individual generators

Lemma 6.3. Ito's formula for Brownian motion with drift

Let $f(x)$ denote an arbitrary twice differentiable function applied to a Brownian motion with drift $X_t = \mu t + \sigma B_t$. Then, the expected value of the function after a small time interval may be approximated by:

$$\mathbb{E}[f(X_{t+h})/X_t = x] \approx f(x) + hGf(x)$$

where

$$(Gf)(x) = \frac{\sigma^2}{2} f''(x) + \mu f'(x) \tag{33}$$

Proof sketch: The key point is again shoing that the differential of "infinitesimal random walk with vanishing drift" may be approximated by:

$$\mathbb{E}[d\ f(S_h(t)/S_h(t) = x] = pf(x+D) + qf(x-D) - f(x) \approx f(x) + h(Gf)(x),$$

which is established in an Exercise below.

Note: Ito's formula may be also derived by a method which uses directly the properties of Brownian motion, to be covered in a later section.

Summary: There are three approaches for working with Brownian motion with drift X_t :

1. $X_t = \mu t + \sigma B_t$, which implies that $\mathbb{E}X_t = \mu t, \operatorname{Var} X_t = \sigma^2 t$.
2. $X_t \approx S_h(t) = \sum_1^{\lfloor t/h \rfloor} X_i$, where the "random walk with infinesitemal bias" has increments $X_i = \pm D$ which gpo up/down with probabilities $p, q = \frac{1}{2} \pm \frac{\mu}{2\sigma^2} D$, and step size satisfying $D^2 = \sigma^2 h$.
3. $\mathbb{E}f((X_{t+h})/X_t = x] \approx f(x) + hGf(x)$ where $(Gf)(x) = \frac{\sigma^2}{2} f''(x) + \mu f'(x)$.

The connection between one and two and two and three are established in the exercise 2a0 and 2b) below, respectively:

Exercise 6.2

a) Show that as the infinitesimal random walk with vanishing bias and Brownian motion with drift have the same expectation and variance, i.e. as $h \to 0$, we have $\mathbb{E}S_h(t) \to \mu t$ and $\operatorname{Var} S_h(t) \to \sigma^2 t$, respectively.

b) Show that for the infinitesimal random walk with vanishing bias $pf(x+D) + qf(x-D) - f(x) \approx h(Gf)x$ where $(Gf)(x) = \frac{\sigma^2}{2} f''(x) + \mu f'(x)$.

Solution 6.2

b) Using the Taylor expansion, we find that for D small we have

$$pf(x+D) + qf(x-D) - f(x) =$$
$$p(f(x) + Df'(x) + \frac{D^2}{2} f''(x) + ...) + q(f(x) - Df'(x) + \frac{D^2}{2} f''(x) - ...) - f(x) =$$
$$(p-q)Df'(x) + D^2 \frac{f''(x)}{2} = h\left(\mu f'(x) + \frac{\sigma^2}{2} f''(x)\right) = h(Gf)(x)$$

where the last steps used the scaling relation $D^2 = \sigma^2 \, h$ and the relation $(p-q) = \frac{\mu}{\sigma^2} D$.

Note: While the first derivative was enough by itself in standard deterministic calculus to give a good approximation of the differential of a function, the addition of the uncertain and "wiggly" Brownian motion leads to the necessity of using also the second derivative.

As illustrated in Exercise 2 below, the differential $pf(x+D) + qf(x-D) - f(x)$ approximated by Ito's lemma appears in various problems involving functionals applied to infinitesimal random walk with vanishing bias. As a consequence, these problems maybe all transformed by Ito's lemma into differential equations involving the differential operator

$$(Gf)(x) = \frac{\sigma^2}{2} f''(x)$$

Exercise 6.3 Differential equations for infinitesimal random walk with vanishing bias a) Formulate the systems of equations of Exercise 1 a)-c) for the "infinitesimal random walk with vanishing bias" $S_h(t)$. Using Ito's approximation $pf(x+D)+qf(x-D)-f(x) \approx f(x) + h(Gf)(x)$, find the differential equations obtained in the limit as $D, h \to 0$ such that $D^2 = \sigma^2 h$.

b) Solve the differential equations in the case $\mu = 0$ (symmetric random walk).

Solution 6.3 a) With the random walk evolving by steps of size D taken after intervals of time of size h, we may check that the "difference operator", i.e. the homogeneous part appearing in all the problems is

$$pf(x+D) + qf(x-D) - f(x).$$

As shown above, after a Taylor expansion and under the scaling assumption $D^2 \approx \sigma^2 h$ this is approximately

$$pf(x+D) + qf(x-D) - f(x) \approx h(\mu f'(x) + \frac{\sigma^2}{2} f''(x) = h(Gf)(x)). \tag{34}$$

After dividing by h and taking limits, this leads respectively to the equations:

1. $(Gf)(x) = 0$ for the exit probabilities and the expected final value of problems 1a), 1b).
2. $(Gf)(x) + 1 = 0$ for the expected exit time problem 1c).

where the operator is the second order differential operator $(Gf)(x) = \frac{\sigma^2}{2} f''(x)$. This operator will appear in all problems involving symmetric Brownian motion with variance σ^2.

b) The solutions of the differential equations are similar, but easier than those of the corresonding difference equations.

For example, we find that the probability of exiting through K when $p = q = 1/2$ is

$$p_x = \mathbb{P}\{X_T = K\} = \mathbb{P}\{T_K < T_0\} = \frac{x}{K}.$$

6.2 Differential equations used in mathematical finance

In the following subsections we discuss the main types of problems encountered in mathematical finance and illustrate the point that when the model is Brownian motion with drift, the solution of each of them reduces to solving a differential equation involving the same associated operator G (33). The various problems differ only by their nonhomogeneous term and boundary conditions.

The two most frequently used types of problems are:

- **Expected present value of final payments** $f(x) = \mathbb{E}_x e^{-rT} h(X_T)$ satisfying:

  $$Gf(x) - rf(x) = 0$$

  with $f = h$ on the boundary.

- **Expected present value of continuos payment flows** $f(x) = E_x[\int_0^T e^{-rt} c(X(t))]\, dt$ which satisfy:

  $$(Gf)(x) - rf(x) + c(x) = 0$$

  with $f = 0$ on the boundary.

We will always assume in the problems above that T is a random time, either the hitting time of some barrier, or an independent exponentially distributed random time. The reason is that while these problems have also versions with fixed expiration time $T = t$, which are in fact more common in applications, the answer of the fixed time problems depends essentially on the remaining time until expiration, while our simplified problems depend only on the starting position. Thus, we are led to solving ordinary differential equations, while the problems with fixed time lead partial differential equations, which can typically only be solved numerically.

As illustrated in the exercise below, the advantage of working with the continuous time models as opposed to their random walk analogues is that analytically, differential equations are easier to solve than the analog difference equations.

Exercise 6.4 Find the "risk neutral" present value of a double barrier binary which expires either "in" at the crossing of a lower barrier L or "out" at the crossing of an upper barrier U:

$$f(S) = \mathbb{E}^*_{S_t=S} e^{-r(T-t)} I_{\{} S_T = L\}$$

where $S_t = S_0 e^{g\,t+\sigma B_t}$ is "risk neutralized" exponential Brownian motion with $g = r - \frac{\sigma^2}{2}$.

Solution 6.4 We make first a change of variables $S_t = S_0 e^{x_t}$ where $x_t = gt + \sigma B_t$ is Brownian motion with drift, so we can apply Ito's lemma for this process. The values of the x_t process corresponding to the barriers and the initial position, l, u, x are given thus by: $l = \log(L/S_0), u = \log(U/S_0)$, $x = \log(S/S_0)$ (they are obtained by plugging the corresponding value of the process S_t in the "master" formula $x_t = \log(S_t/S_0)$).The functional becomes $f(x) = \mathbb{E}_{x_0=x} e^{-rT} I_{\{} X_T = \log(L/S_0)\}$ where $T = \min\{T_l, T_u\}$ and are all obtained from the change of scale $S = S_0 e^x$, This is a discounted final payment problem, and the final payoff is either 1 or 0 and so we must solve the associated differential equation:

$$\frac{\sigma^2}{2} f''(x) + g f'(x) - r f(x) = 0$$
$$f(l) = 1$$
$$f(u) = 0.$$

Letting a_1, a_2 enote the roots of the associated equation $\frac{\sigma^2}{2} a^2 + (r - \frac{\sigma^2}{2}) a - r = 0$ we find that $a_1 = \frac{-2r}{\sigma^2}$, $a_2 = 1$ and

$$f(x) = \frac{e^{a_1(x-u)} - e^{(x-u)}}{e^{a_1(l-u)} - e^{(l-u)}}$$

In terms of the original variables this becomes

$$f(S) = \frac{(S/U)^{a_1} - (S/U)}{(L/U)^{a_1} - (L/U)}$$

6.2.1 Expected final payments

Definition: An expected final payment functions is a function of the form:

$$f(x) = \mathbb{E}_x h(X_T)$$

where $h(x)$ denotes some final payment.

The time T could be taken as fixed, like in the problem of European options; however, this leads to more complicated (partial) differential equations. For this reason we will consider only the mathematically simpler case when T is a random time, like for example the hitting time of a boundary.

The most typical example, discussed below, is that of **escape probabilities**, in which the boundary is divided in a "good" part A where the payment is 1 and a "bad" part where the payment is 0. The payment is thus $h(x) = I_A(x)$.

Lemma 6.4. *The expected final payment problem leads to the equation*

$$Gf(x) = 0$$

with $f = h$ on the boundary.

Proof: We may either consider the approximating discrete problem $pf(x + D) + qf(x - D) - f(x) = 0$ and use its Taylor approximation $pf(x + D) + qf(x - D) - f(x) = h(Gf)(x)$ or work directly in continuous time conditioning on the value after a small time interval h as follows:

$$f(x) = \mathbb{E}_x h(X_T) = \mathbb{E}_x \mathbb{E}_{X_h} h(X_T) = \mathbb{E}_x f(X_h) = f(x) + h(Gf)(x)$$

where the last step follows from Ito's lemma. Cancelling $f(x)$ and dividing by h yields now the differential equation. The boundary conditions are obvious if we are already at the boundary, the expected payment must equal the final payoff).

Example 1: Escape probability from an an interval $[L, K]$

We compute now $f_x = \mathbb{P}_x\{X(T) = K\}$ that a Brownian motion with drift $X(t)$ starting at x will exit an interval $[L, K]$ through the upper endpoint.

We consider first the associated discrete biased random walk. The difference equation satisfied by the probability $p(x)$

$$p_x = P_x\{S_h(T) = K\}$$

for the associated discrete biased random walk are:

$$\begin{aligned} p(x) &= p\,p(x+D) + q\,p(x-D) \\ p(K) &= 1 \\ p(0) &= 0 \end{aligned}$$

Using Taylor expansions carefully, we find that $p\,p(x+D)+q\,p(x-D)-p(x) \approx h\,(\frac{\sigma^2}{2}p''(x)+\mu p'(x))$ and so the system becomes

$$\begin{aligned} \frac{\sigma^2}{2}p''(x) + \mu\,p'(x) = 0 \\ p(K) = 1 \\ p(0) = 0 \end{aligned}$$

Note: As expected, this is of the form

$$\begin{aligned} (Gp)(x) = 0 \\ p(K) = 1 \\ p(0) = 0 \end{aligned}$$

where $(Gp)(x) = \frac{\sigma^2}{2}p''(x) + \mu\,p'(x)$.

The general solution of the equation above is

$$p(x) = A_1\,(\exp{(-\frac{2\mu}{\sigma^2}x)} + A_2)$$

which after using the boundary conditions yields

$$p(x) = \frac{\exp{(-\frac{2\mu}{\sigma^2}x)} - 1}{\exp{(-\frac{2\mu}{\sigma^2}K)} - 1} = \frac{f(x) - 1}{f(K) - 1}$$

where $f(x) = \exp{(-\frac{2\mu}{\sigma^2}x)}$.

6.2.2 Expected total continuous payments

Definition: Expected continuous payment functions are functions of the form

$$f(x) = E_x[\int_0^T c(X(t))]\ dt$$

where $c(x)$ represent the rate of payment (cost) per unit time.

Note: This represents the continuous time limit of the discrete costs $\sum_{i=0}^{[T/h]} c(X(i\ h))h$ when $h \to 0$.

We will work for convenience mostly with the case when T is a random time, for example the hitting time of some boundary.

Lemma 6.5. *An expected continuous payment function $f(x)$ must satisfy:*

$$(Gf)(x) + c(x) = 0$$

with $f = 0$ on the boundary.

We may either proof this by considering an approximating discrete problem and using the Taylor approximation (**??**) or by working directly in continuous time, conditioning on the value after a small time interval h.

Proof 1: For the random walk which approximates Brownian motion with drift we find by conditioning after one step the inhomogeneous equation:

$$f(x) = p\ f(x + D) + q\ f(x - D) + h\ c(x)$$

which after using the Taylor approximation (**??**) reduces to:

$$h\ (\frac{\sigma^2}{2} f''(x) + \mu\ f'(x)) +\ c(x)) = 0.$$

yielding in the limit: $(Gf)(x) + c(x) = 0$ where Gf denotes the Brownian motion with drift operator $\frac{\sigma^2}{2} f''(x) + \mu\ f'(x)$.

Proof 2: Working directly in continuous time, we condition on the value after a small time interval h.

$$f(x) = \mathbb{E}_x \int_0^T c(X_t)dt =$$
$$\mathbb{E}_x \int_0^h c(X_t)dt + \mathbb{E}_x \int_h^T c(X_t)dt \approx c(x)h + \mathbb{E}_x \mathbb{E}_{X_h} \int_h^T c(X_t)dt =$$
$$c(x)h + \mathbb{E}_x f(X_h) \approx c(x)h + f(x) + hGf(x)$$

where the last step used Ito's lemma.

Cancelling $f(x)$ and dividing by h yields now the differential equation. The boundary conditions are obvious (the total cost over a 0 length interval is 0).

scr **Note:** The second proof rests only on the Ito's lemma previously obtained for Brownian motion with drift. Ito's lemma is also true for a more general class of processes called **diffusions** (These processes, discussed in the next section, are basically Brownian motions for which the drift $\mu(x)$ and volatility $\sigma(x)$ are allowed to depend on the current position). Hence, the lemma above also holds in that more general framework.

One example is when $c(x) = 1$ identically, in which case $f(x)$ becomes the expected exit time $t(x) = \mathbb{E}_x T$.

Example 2: The expected exit time from (L, K) for a Brownian motion with drift

We have found in Exercise 2 that the expected time $t(x)$ when starting at x until exiting (L, K) for Brownian motion without drift is the solution of

$$\begin{aligned} (Gt)(x) + 1 &= 0 \\ t(K) &= 0 \\ t(L) &= 0 \end{aligned}$$

where $(Gt)(x) = \frac{\sigma^2}{2} t''(x)$ is the generator of Brownian motion without drift. It is easy to guess (and check) that the procedure of considering the biased random walk approximation and passing to the limit will yield the differential equation:

$$\begin{aligned} (Gt)(x) + 1 &= 0 \\ t(K) &= 0 \\ t(L) &= 0 \end{aligned}$$

where $(Gt)(x) = \frac{\sigma^2}{2} t''(x) + \mu\, t'(x)$ is now the generator of Brownian motion with drift. Thus, we must solve:

$$\begin{aligned} \frac{\sigma^2}{2} t''(x) + \mu\, t'(x) + 1 &= 0 \\ t(K) &= 0 \\ t(L) &= 0 \end{aligned}$$

Solution: The homogeneous solution $p(x) = A_1\ (\exp(-\frac{2\mu}{\sigma^2} x) + A_2)$ was obtained in the previous problem.

Our initial guess for a particular solution is $t_p(x) = B$, but we have to modify it to $t_p(x) = Bx$, since constants satisfy the homogeneous problem. We find that $B = -\frac{1}{\mu}$ and $t_p(x) = -\frac{x}{\mu}$.

The general solution is thus of the form $t(x) = -\frac{x}{\mu} + a\,(\exp{(-\frac{2\mu}{\sigma^2}x)} + b)$. The boundary conditions yield

$$\begin{aligned} t(x) &= -\frac{x-L}{\mu} + \frac{K}{\mu}\frac{\exp{(-\frac{2\mu}{\sigma^2}x)} - \exp{(-\frac{2\mu}{\sigma^2}L)}}{\exp{(-\frac{2\mu}{\sigma^2}K)} - \exp{(-\frac{2\mu}{\sigma^2}L)}} \\ &= t_p(x) - t_p(K)\frac{f(x)-f(L)}{f(K)-f(L)} \end{aligned}$$

where t_p denotes the particular solution which is 0 at $x = L$ and $f(x) = \exp{(-\frac{2\mu}{\sigma^2}x)}$ is the nonconstant homogeneous solution.

Example 3: The expected total inventory cost problem over $T = \min(T_0, T_K)$ until exiting $(0, K)$ for Brownian motion with drift. The cost

$$i(x) = \mathbb{E}_x \int_0^T X_t dt$$

must satisfy:

$$\begin{aligned} (Gi)(x) + x &= 0 \\ i(K) &= 0 \\ i(0) &= 0 \end{aligned}$$

Plugging the generator of Brownian motion with drift leads to:

$$\begin{aligned} \frac{\sigma^2}{2} i''(x) + \mu\, i'(x) + x &= 0 \\ i(K) &= 0 \\ i(0) &= 0 \end{aligned}$$

The particular solution (obtained by modifying once the initial trial $Ax+B$ is of the form $i_p(x) = Ax^2 + Bx$. By the method of undetermined coefficients we find that $i_p(x) = \frac{x^2}{-2\mu} + x\frac{\sigma^2}{2\mu^2}$.

Finally, as in Example 1, we find from the boundary conditions that:

$$i(x) = i_p(x) - i_p(K)\frac{f(x)-f(0)}{f(K)-f(0)}$$

where $f(x) = \exp{(-\frac{2\mu}{\sigma^2}x)}$ is the nonconstant homogeneous solution and i_p denotes the particular solution which is 0 at $x = 0$.

6.2.3 Expected discounted final payments

Definition: Expected discounted final payment functions are functions $f(x)$ of the form:

$$f(x) = \mathbb{E}_x e^{-rT} h(X_T)$$

Lemma 6.6. *Expected discounted final payment functions satisfy:*

$$Gf(x) - rf(x) = 0$$

with $f = h$ on the boundary.

Proof:

$$\begin{aligned} f(x) &= \mathbb{E}_x e^{-rT} h(X_T) = e^{-rh} \mathbb{E}_x e^{-r(T-h)} h(X_T) \\ &= e^{-rh} \mathbb{E}_x \mathbb{E}_{X_h} e^{-r(T-h)} h(X_T) \approx (1 - rh) \mathbb{E}_x f(X_h) \\ &\approx (1 - r\,h)\,(f(x) + hGf(x)) \approx f(x) + h((Gf(x) - r\,f(x)) \end{aligned}$$

where the last steps consist in using Ito's lemma and ignoring the h^2 term.

A typical example is the present value of a double barrier option which "kicks in" if a lower barrier L is reached and expires if an upper barrier U is reached.

6.2.4 Expected present value of continuous payment flows

Definition: Expected present value of continuous payment functions are functions of the form

$$f(x) = E_x[\int_0^T e^{-rt} c(X(t))]\ dt$$

where $c(x)$ represent the rate of payment (cost) per unit time.

Lemma 6.7. *An expected continuous payment function $f(x)$ must satisfy:*

$$(Gf)(x) - rf(x) + c(x) = 0$$

with $f = 0$ on the boundary.

A typical example is the present value of a fixed dividend k, to be payed until the asset hits an upper barrier u.

6.3 Summary

Two general types of functionals are most common in problems of valuation of barrier options exercised at some stopping time T : expected final payoffs and expected continuous payments. Each has also a version with discounting and they may be all incorporated as particular cases in a general formula called the Feynman-Kac formula. Each problem leads to a certain associated ordinary differential equation:

- **Expectation of final payoffs** $f(x) = \mathbb{E}_x h(X_T)$ satisfying:

$$Gf(x) = 0$$

with $f = h$ on the boundary.

- **Expected total continuous payments** $f(x) = \mathbb{E}_x \int_{t=1}^{T} c(X_t) dt$ satisfying:

$$(Gf)(x) + c(x) = 0$$

with $f = 0$ on the boundary.

- **Expected discounted final payoffs** $f(x) = \mathbb{E}_x e^{-rT} h(X_T)$ satisfying:

$$Gf(x) - rf(x) = 0$$

with $f = h$ on the boundary.

- **Expected present value of continuous payment flows** $f(x) = E_x[\int_0^T e^{-rt} c(X(t))]\, dt$ which satisfy:

$$(Gf)(x) - rf(x) + c(x) = 0$$

with $f = 0$ on the boundary.

- **The Feynman-Kac formula**
$f(x) = E_x[e^{-\int_0^T r(X_u)du} h(X_T) + \int_0^T e^{-\int_0^t r(X_u)du} c(X(t))]\, dt$ satisfies:

$$(Gf)(x) - r(x)f(x) + c(x) = 0$$

with $f = h$ on the boundary.

Note that the general Feynman-Kac formula allows a non constant random discount rate $r(X_u)$, which is more realistic; for this to be really useful in applications however, X_t has to be interpreted as some vector composed of several indices which determine (deterministically) the interest rate. Thus, X_t is not to be interpreted as a single asset anymore, but as some "multifactor" which drives the economy.

Each of these problems has also a version with fixed expiration time, which involves however an extra variable (the remaining time until expiration) and lead to solving partial differential equations. For example, the **present value of final fixed time payment** functional:

$$f(x, t) = \mathbb{E}_{x,t} e^{-rt} h(X_t)$$

which satisfies

$$\frac{\partial}{\partial t} f + (Gf) - rf = 0$$

with $f(x, t) = h(x)$ at the final time.

The last and (more commonly used in practice) example is different of the previous four in that the expectation depends on two variables: the starting point x and the current time

t (or rather on the remaining time $T-t$. This leads to **partial differential equations**, which look similar to the ordinary differential equations previously discussed, but are actually considerably harder to solve.

This partial differential equation could be derived in principle by the usual method of starting with a discrete approximation, conditioning on one step and taking limits, or by starting with the partial differential equation satisfied by the density of standard Brownian motion:

** **Extra problem** a) Show that the normal density $p(t,x) = \frac{\exp\left(-\frac{x^2}{2\sigma^2 t}\right)}{\sqrt{2\pi\sigma^2 t}}$ satisfies the partial differential equation

$$\frac{\partial}{\partial t}p + \frac{\partial^2}{\partial x^2}p = 0$$

b) Conclude that $g(x,t) = \mathbb{E}_{x,t}h(X_T) = \int p(t, x-y)h(y)dy$ satisfies the same partial differential equation.

c) Derive the partial differential equation satisfied by $f(x,t) = e^{-rt}g(x,t)$.

While we need to be aware of the existence of these various PDE's (partial differential equations) for the expected present values of options, because of their crucial role in solving the problems of mathematical finance numerically via the use of computers, we will not use them, since numerical approaches are not featured in this course.

6.4 Exercises

Exercise 6.5 If $B(t)$ is standard Brownian motion on the interval $[L, K]$, find the probability of exiting through the lower barrier $p_x = \mathbb{P}\{T_L \leq T_K\}$.

Exercise 6.6 If $B(t)$ is standard Brownian motion, find $\mathbb{P}_0\{T_1 < T_{-1} < T_2\}$.

Exercise 6.7 Prove Lemma 6.6, starting from the difference equation for the analog problem for random walk with vanishing bias and taking the limit as $D, h \to \infty$.

Exercise 6.8 Formulate and solve a differential equation for the expected exit time from (L, ∞) for a Brownian motion with drift $\mu < 0$. Compare the result with the result obtained in the deterministic case $X_t = \mu t$.

Exercise 6.9 "Bankruptcy time" Suppose that the evolution of an asset follows the lognormal model S_t = exponential Brownian motionwith a negative parameter g. The asset will be liquidated at the stopping time $T_a = \inf\{t : S_t = a\}$ when its value reduces to e^{-a}, where a is a number less than 1. Find the expected value of the time T_a.

Exercise 6.10 Find the "risk neutral" present value of a perpetual (no time limit) "down and in" lower barrier binary:

$$f(S) = \mathbb{E}^*_S e^{-rT_L} I_{\{}T_L < \infty\}$$

where $S_t = S_0 e^{g\,t + \sigma B_t}$ is "risk neutralized" exponential Brownian motion with $g = r - \frac{\sigma^2}{2}$.

Exercise 6.11 Perpetual American put: a) The perpetual barrier put: Find the "risk neutral" present value of a "perpetual" put-barrier option with final payment $(K - S_T)_+ = K - L$, to be exercised the first time when the stock price crosses a lower barrier L.

b) Suppose the buyer of the option can choose the barrier L at which to exercise (this is called an American option). What is the optimal choice for the barrier?

Exercise 6.12 Find the present value of a fixed dividend k, to be payed until the asset hits an upper barrier u:

$$f(x) = \mathbb{E}_x \int_0^T e^{-rt} k dt$$

where $X_t = S_0 e^{g\,t + \sigma B_t}$ is exponential Bronian motion and $T = T_u$ is the hitting time of the barrier.

Exercise 6.13 a) Formulate and solve a differential equation for the expected total inventory cost until hitting 0 for a deterministic case $X_t = \mu t$ with $\mu < 0$.

b) Formulate and solve a differential equation for the expected total inventory cost until hitting 0 for a Brownian motion with drift $\mu < 0$. Compare the result with the result obtained in the deterministic case $X_t = \mu t$.

6.5 Solutions

Solution 6.5 We need a formula for $\mathbb{P}_x\{T_L < T_K\}$ for a general state space $[L, K]$. This is a generalization of the previous "probability of hitting K before 0 problem. The differential equation is

$$\begin{aligned} Gf &= 0 \\ f(L) &= 1 \\ f(K) &= 0 \end{aligned}$$

where $(Gf)(x) = \frac{1}{2} f''(x)$

and the solution is $\frac{K-x}{K-L}$.

Solution 6.6 By the general rule for conditional probabilities, $\mathbb{P}_0\{T_1 < T_{-1} < T_2\} = \mathbb{P}_0\{T_1 < T_{-1}\}\mathbb{P}_1\{T_{-1} < T_2/$ conditional on $\{T_1 < T_{-1}\}\}$. We can however ignore the conditioning since $B(t)$ is a Markov process. Furthemore, by the stationarity of increments, $\mathbb{P}_1\{T_{-1} < T_2\} = \mathbb{P}_0\{T_{-2} < T_1\}$.

Applying the result of the previous exercise twice, we find

$$\mathbb{P}_0\{T_1 < T_{-1} < T_2\} = \frac{1}{2}\frac{1}{3} = \frac{1}{6}.$$

Solution 6.7 The expected present value $f(x) = \mathbb{E}_x e^{-r\tau} h(X_\tau)$ satisfies:

$$f(x) = e^{-rh}(pf(x+D) + qf(x-D)$$

The boundary conditions are $f(K) = h(K), f(L) = h(L)$.

After a Taylor expansion, using the scaling relation $D^2 = \sigma^2 h$ and the fact that $p, q = \frac{1}{2} \pm \frac{\mu}{2\sigma^2} D$ satisfy $p - q = \frac{\mu}{\sigma^2} D$, we find:

$$\begin{aligned} f(x) &\approx (1-rh)\left(p(f(x) + Df'(x) + \frac{D^2}{2} f''(x) + ...) + q(f(x) - Df'(x) + \frac{D^2}{2} f''(x) - ...)\right) \\ &\approx f(x) + h\left(\mu f'(x) + \frac{\sigma^2}{2} f''(x)\right) = h(Gf)(x) \end{aligned}$$

Solution 6.8

$$\begin{aligned} \frac{\sigma^2}{2} t''(x) + \mu\, t'(x) + 1 &= 0 \\ t(\infty) \text{ is not "exponentially blowing up"} & \\ t(L) &= 0 \end{aligned}$$

Solving this, or passing to the limit $K \to \infty$ in the result of Example 2 (using $\mu < 0$) we find $t(x) = -\frac{x-L}{\mu}$.

Interestingly, this does not depend on the volatility parameter σ and is in fact the same answer we obtain in the deterministic case $\sigma = 0$, i.e. when $X_t = \mu t$.

Solution 6.9 The **Expected Bankruptcy time**

Solution 6.10 "Down and In" binary: We make first a change of variables $S_t = S_0 e^{x_t}$ where $x_t = gt + \sigma B_t$ is Brownian motion with drift, so we can apply Ito's lemma for this process. The functional becomes $f(x) = \mathbb{E}_{X_0=x} e^{-rT} I_{\{T_l} < \infty\}$ where $l = \log(L/S_0)$ and $x = \log(S/S_0)$. This is a discounted final payment problem, and so we must solve the associated differential equation:

$$\begin{aligned} \frac{\sigma^2}{2} f''(x) + g f'(x) - r f(x) &= 0 \\ f(l) &= 1 \\ f(\infty) &= 0. \end{aligned}$$

This homogeneous equation has two exponential solutions $e^{a_1 x}, e^{a_2 x}$ where a_1, a_2 are the roots of the equation $\frac{\sigma^2}{2} a^2 + (r - \frac{\sigma^2}{2}) a - r = (\frac{\sigma^2}{2} a + r)(a-1) = 0$. The solution e^x blows up at ∞ so it can't be used. In conclusion, $f(x) = e^{-\frac{2r}{\sigma^2}(x-l)}$. Back to the original variables we find

$$f(S) = (L/S)^{\frac{2r}{\sigma^2}}$$

Note: an alternative derivation is via the Wald martingale; note that $a_2 = -\frac{2r}{\sigma^2}$ is precisely the negative root of Wald's equation $c(a) = r$.

Solution 6.11 Perpetual American put: a) The value of the perpetual put barrier is the value of a "down and in" binary (see the previous Exercise) multiplied by $K - L$.

$$f(S) = (K - L)(L/S)^{\frac{2r}{\sigma^2}}$$

b) The optimal exercise barrier is found by setting the derivative with respect to L equal to 0. ....

We note that it does not depend on the initial starting point S (this is due to the Markov property of the asset model).

Solution 6.12 Dividend until an upper barrier:

Solution 6.13

a) The deterministic discrete approximating equation is $f(x) = hc(x) + f(x + \mu\, h)$ differs from the random walk one by containing only one possible for X at time $t+h$, which in the limit results in a first order equation, as opposed to second order in the random case.

$$\begin{aligned} \mu\, f'(x) + c(x) &= 0 \\ f(0) &= 0 \end{aligned}$$

If, say, $c(x) = x$, the solution is $\frac{x^2}{-2\mu}$, as can be easily checked directly (the emptying time is $\frac{x}{-\mu}$ and the cost is the area of a triangle with sides x, $\frac{x}{-\mu}$.

It is interesting to compare the stochastic inventory problem with the deterministic case (corresponding to $\sigma = 0$) when $X(t) = x + \mu t$. For large x, the dominant first term in the solution above is independent of the uncertainty parameter σ and coincides in fact with the integral of the deterministic emptying solution $x + \mu t$. For small x however σ is important.

7 Financial variations on Black Scholes

In this section we review the Black Scholes theory and discuss some extensions.

We start by recalling the valuation formula for the **forward** on an asset

$$F = S_0$$

and the valuation formula for the **call option**, which, taking into account the previous identity, will be written this time as:

$$C = F\Phi\left(\frac{\log(\frac{F}{\tilde{K_T}})}{\sqrt{V_T}} + \frac{\sqrt{V_T}}{2}\right) - \tilde{K_T}\Phi\left(\frac{\log(\frac{F}{\tilde{K_T}})}{\sqrt{V_T}} - \frac{\sqrt{V_T}}{2}\right)$$

The point of this new formulation is that it turns out to be also true in other aplications discussed below, in which the forward value isn't equal anymore to the current asset value. More specifically, this formula continues to apply in the case of assets on a foreign exchange, with the forward price of a foreign stock being given by

$$F = S_0 E_0$$

where E_0 denotes the initial exchange rate, and in the case of options on foreign currencies with interest rate y and current exchange rate E_0 or on assets yielding continuous dividends at rate δ, in which case the forward prices are:

$$F = E_0 e^{-yT} \qquad \text{and} \qquad F = S_0 e^{-\delta T}$$

7.1 Assets on a foreign exchange

A british investor invests in a stock on the Japanese exchange. It is assumed that the stock's value evolves as geometric Brownian motion $S_t = e^{X_t}$, where $X_t = g_1 t + \sigma_1 B_1(t)$ is Brownian motion with drift g_1 and variance σ_1^2. Suppose that the interest rates in Britain is r, and that the exchange rate is such that 1 yen will be worth at time t $E_t = E_0 e^{g_2 t + \sigma_2 B_2(t)}$ where $B_1(t), B_2(t)$ are Brownian motions with correlation ρt.

To figure out the risk neutral sterling value of various financial derivatives it is enough to note that the sterling value of a foreign asset $S_t E_t$ is itself an exponential Brownian motion given by $S_t E_t = e^{gt+\sigma B_t}$ where $g = g_1 + g_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho\sigma_1\sigma_2$ (the latter follows from the addition formula for correlated Gaussian random variables). In conclusion, the usual formulas for call, put, etc. apply, where however S_0 is replaced by $S_0 E_0$; the latter is precisely the forward value of S_T for the British investor.

A somewhat unusual barrier option is described in the following exercise:

Exercise 7.1 Exchange adjusted barrier option

1. Let $a > 0$ and let $T_a = \inf\{t : \; X_t = a\}$ where X_t is standard Brownian motion. Formulate and solve a differential equation for the Laplace transform of T_l

$$f(x) = \mathbb{E}_{X_0=x}[e^{-rT_l}].$$

Conclude that the Laplace transform of T_l starting from $X_0 = 0$ is:

$$\mathbb{E}_0[e^{-rT_l}] = e^{-l\sqrt{2r}}, \quad r > 0.$$

2. Using the previous answer (or otherwise) find the expectation $\mathbb{E}_0 T_l$.

   What is the Laplace transform $\mathbb{E}_0[e^{-rT_l}]$ when $r < 0$? .

   A british investor buys one unit of stock on the Japanese exchange, whose foreign currency price evolves as geometric Brownian motion $S_t = S_0 e^{X_t}$, where X_t is **standard Brownian motion** with $X_0 = 0$. Suppose that the interest rates in both countries are 0, and that the exchange rate is such that 1 yen will be worth at time t $E_t = E_0 e^{-\mu t + \sigma B_t}$ where B_t is a Brownian motion **independent** of X_t and $\mu > 0$ (a strong pound).

   The investor decides to sell the stock at the time T when the foreign currency price $S_0 e^{X_t}$ reaches a pre-determined level $S_0 e^l > S_0$ (so $l > 0$). The exercise illustrates the fact that the profits of this strategy, which disregards the real value $S_t E_t$ of the asset for the british investor, will be usually eroded by the strong pound.

3. What is the value of the investment at time t in the investor's currency (in pounds). Find an expression for the expected final payoff per pound invested.

4. Conditioning on the value of T, find the expected final value of the (exchange adjusted) investment if the parameters μ and σ satisfy $2\mu > \sigma^2$, and determine under what conditions the profit is negative and positive, respectively.

5. Explain what happens if $2\mu < \sigma^2$ and comment.

Solution 7.1

1. This problem is already expressed in terms of a standard Brownian motion X_t. It may be viewed as computing the expected discounted value of a sure payoff of 1 achieved when the boundary l is hit by X_t. The differential equation is:

$$\begin{aligned} \frac{1}{2} f''(x) - r f(x) &= 0 \\ f(l) &= 1 \\ f(-\infty) &= 0 \end{aligned}$$

   with solution $f(r, x) = e^{(x-l)\sqrt{2r}}$. Note this only works when $r > 0$.

   For $x = 0$ we find $f(r, 0) = e^{-l\sqrt{2r}}$

2. Differentiating the Laplace transform $f(r,0)$ of T_a with respect to r and plugging $r=0$ we find $\mathbb{E}_0 T_l = -\frac{\partial}{\partial r} f = \infty$.

   When $r<0$ we expect $E[e^{-rT_l}] = \infty$ since T_l and so $-rT_l$ is likely to be very large, or since the discounting is negative and so the value keeps increasing, for a time which has infinite expectation.

   More formally, using $e^x > 1+x$ we see that for any l and $r<0$ we have $E[e^{-rT_l}] > 1 + (-r\mathbb{E}_0 T_l) = \infty$.

3. The (exchange adjusted) value of the investment in the investor's currency is given by: $E_0 e^{-\mu t + \sigma B_t} e^{X_t}$. The expected final payoff per pound invested is $\mathbb{E}[e^l e^{-\mu T + \sigma B_T}]$, (where we put $T = T_l$).

4. Let $f_T(t)dt$ denote the stopping time's density. Conditioning on $T=t$ we find:

$$\mathbb{E}[e^l e^{-\mu T + \sigma B_T}] = e^l \int_t f_T(t)\, dt\, e^{-\mu t} \mathbb{E}[e^{\sigma B_T} \;/T = t]$$

$$= \quad e^l \int_t f_T(t)\, dt\, e^{-\mu t} e^{\frac{\sigma^2 t}{2}} \qquad \text{since } T \quad \text{is} \quad \text{independent of } B_t$$

$$= \quad e^l \int_t f_T(t)\, dt\, e^{-(\mu - \frac{\sigma^2}{2})t} = e^l \mathbb{E} e^{-rT_a}$$

   where we put $r = \mu - \frac{\sigma^2}{2}$. If $r>0$ (which happens if $2\mu > \sigma^2$) we may conclude by part a) that the expected final value is

$$e^l e^{-l\sqrt{2\lambda}} = e^{l(1-\sqrt{2\lambda}}$$

   The profit has negative expectation iff the expected final payoff per pound will be smaller than 1. This happens iff the exponent of l $1-\sqrt{2r}$ is negative, i.e. iff $2r = 2\mu - \sigma^2 > 1$, or $2\mu > \sigma^2 + 1$. In this case the strong pound is expected to erode the investor's profits!

   The profit has positive expectation iff the exponent is positive, i.e. iff $\sigma^2 < 2\mu < \sigma^2 + 1$.

5. If $0 < 2\mu < \sigma^2$ then we have infinite expected profits by (b).

7.2 Options on currency and on assets yielding dividends

Consider first the case of a forward on a unit of foreign currency, say yens, which evolve as $Y_t = Y_0 e^{yt}$. Assume the exchange value is given by exponential Brownian motion$E_t = E_0 e^{\mu t + \sigma B_t}$ of home currency units for each yen (and the home currency brings interest r, which will not be needed here).

It turns out just as in the case of forwards on assets that the best hedging strategy is **static**, namely to buy foreign currency and hold it until expiration. However, since the foreign currency produces interest, to have one unit later it is enough to buy e^{-yT} units now, which will require $E_0 e^{-yT}$ units of the home currency.

In conclusion, the forward value of a unit of foreign currency is

$$F = E_0 e^{-yT}$$

Note the extra discount factor which decreases the current value with respect to the case $y = 0$.

Precisely the same formulas holds in the case of assets which yield continuous dividends at some rate δ. Let S_t denote the value of one stock unit at time t and let $N_t = N_0 e^{\delta t}$ denote the number of stock units at time t for an investor who starts with N_0 stock units (this changes by exponential increase, just as in the case of cash producing fixed interest). The total value of an investment evolves thus as $\tilde{S}_t = N_t S_t = N_0 S_t e^{\delta t}$, which is of the same nature as the total value of a foriegn currency with constant interest. By the same argument of static hedging, this leads to the same valuation for the forward, with the dividend rate δ replacing the interest rate y.

Furthemore, it turns out (we will not prove that) that the value of a call option is still given by

$$C = F\Phi\left(\frac{\log(\frac{F}{\tilde{K}_T})}{\sqrt{V_T}} + \frac{\sqrt{V_T}}{2}\right) - \tilde{K}_T\Phi\left(\frac{\log(\frac{F}{\tilde{K}_T})}{\sqrt{V_T}} - \frac{\sqrt{V_T}}{2}\right)$$

In the next section we consider a more general financial derivative which involves an exchange between two arbitrary assets.

7.3 Exchange options

Definition Given any two assets S_1, S_2 an **exchange option** is defined as the right (but not the obligation) to exchange asset S_2 for asset S_1, or of obtaining the payoff $(S_1 - S_2)_+$.

Theorem 7.1. *The value of an exchange option on two assets described by geometric Brownian motions:*

$$S_i(t) = S_i(0)e^{g_i\, t + \sigma_i B_i(t)},\ i = 1,2$$

where $B_i(t)$ are Brownian motions with correlation ρ is given by:

$$E = F_1\Phi\left(\frac{\log(\frac{F_1}{F_2})}{\sqrt{V_T}} + \frac{\sqrt{V_T}}{2}\right) - F_2\Phi\left(\frac{\log(\frac{F_1}{F_2})}{\sqrt{V_T}} - \frac{\sqrt{V_T}}{2}\right)$$

where the remaining volatility is $V_T = T(\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2)$.

To understand this formula, it is further useful to define the "moneyness" of an exchange by $m_{1,2} = \frac{\log(\frac{F_1}{F_2})}{\sqrt{V_T}}$. The moneyness is positive or negative depending on whether then exchange is currently desirable or not. In terms of the moneyness, the value of the exchange option is given by $E = F_1\Phi(m_{1,2} + \frac{\sqrt{V_T}}{2}) - F_2\Phi(m_{1,2} - \frac{\sqrt{V_T}}{2})$ which may be summarized as:

The option seller should hedge by determining the moneyness of the exchange, and keep positive (negative) proportions of the forward values of the first (second) asset, computed by applying the normal cdf to the moneyness adjusted upward(downward) by a factor proportional to the remaining volatility.

Note: 1) Two examples of exchange options are the call and the put. In the call case, the moneyness is determined by taking the quotient of the present values (or current forward values) F and $\tilde{K_T}$ of the stock unit and of the exercise price. In the put case, the reciprocal quotient is used.

2) When $K = 1$, the forward value of one cash unit is also called a zero coupon bond.

The proof of the theorem above is left as an exercise.

Exercise 7.2 Derive the formula for the risk neutral value $\mathbb{E}e^{-rT}(S_1(T) - S_2(T))_+$ of an **exchange option** on two exponential Brownian motion assets

$$S_i(t) = S_i(0)e^{g_i\,t + \sigma_i B_i(t)},\ i = 1, 2$$

where B_1, B_2 are two Brownian motions with correlation ρ.

Hint: Use the price of the second asset as an artificial currency (called "numeraire"). The "numeraire" price of any asset $S(t)$ is given by $\frac{S(t)}{S_2(t)}$; thus, the "numeraire" price of the second asset is equal to 1 at any time t and in conclusion using "numeraire" puts us in the situation when the second asset is constant. This is however precisely the previously discussed case of a call option in a market with 0 interest rate, for which we may apply the classical Black Scholes formula.

1. The "numeraire" price of the first asset becomes $Y_t = \frac{S_1(t)}{S_2(t)}$ What is the distribution of Y_t?
2. Find the "numeraire" Black Scholes value for exchanging $Y(T)$ by 1, as well as the value of the exchange option $\mathbb{E}e^{-rT}(S_1(T) - S_2(T))_+$ measured in original currency.
3. Find the price of the standard call and put options on a GBM asset in a market with interest rate r and show that they satisfy the "call-put" parity formula

$$C = P + F - \tilde{K}$$

Solution 7.2

1. Y_t is also a geometric Brownian motion

$$Y_t = e^{gt + \sigma B_t}$$

with parameters $g = g_1 - g_2$ and with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

2. The "fair numeraire price" of the exchange option is obtained by plugging $Y_0 = \frac{S_0^1}{S_0^2}$ instead of S_0 and 1 instead of K in the $r = 0$ Black Scholes formula (by risk neutrality, this means assuming that $Y_t = e^{-\frac{\sigma^2}{2}t + \sigma B_t}$) yielding:
$\frac{S_1(0)}{S_2(0)}\Phi(\frac{\frac{S_1(0)}{S_2(0)} + \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}) - \Phi(\frac{\frac{S_1(0)}{S_2(0)} - \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}})$.

The "fair original currency price" at time t is obtained by multiplying with $S_2(t)$. Thus, for the initial value we have to multiply with $S_2(0)$, yielding $S_1(0)\Phi(s) - S_2(0)\Phi(l)$ where $s, l = \frac{\frac{S_1(0)}{S_2(0)} \pm \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}$ are the adjusted moneynesses. Replacing $S_i(0)$ by the forward values F_i yields the final answer.

3. In the case of the call (put) options in markets with non zero interest rate, the cash asset evolves as $S_2(t) = S_0^2 e^{rt}$ ($S_1(t) = S_0^1 e^{rt}$) and its prescribed final value is K. The forward (present) value is thus $S_2(0) = Ke^{-rT}$ ($S_1(0) = Ke^{-rT}$). We get the respective formulas

$$C = S_0\Phi(s) - Ke^{-rT}\Phi(l) \qquad\qquad P = Ke^{-rT}\Phi(-l) - S_0\Phi(-s)$$

Since $\Phi(-x) = 1 - \Phi(x)$ we get the "put-call" parity relation:

$$P = Ke^{-rT} - S_0 + C = C + \tilde{K} - F$$

which has the clear investment interpretation that buying the stock and a put and taking a loan of $\tilde{K}$ is the same as buying a call.

7.4 Exercises

Exercise 7.3

a) Let $l > x$ and let $T_l = \inf\{t : \ X_t = l\}$. Formulate and solve a differential equation for the Laplace transform $f(x) = \mathbb{E}_{X_0=x} e^{-rT_l}$, if X_t is Brownian motion with drift μt and variance $\sigma^2 t$.

b) Using the previous answer (or otherwise) find the expectation $\mathbb{E}_0 T_l$.

c) Solve the analogous problem when $x > l$.

8 Canadian options

There is something missing currently from our picture on how to solve option problems by differential equations, and the hole will be filled by introducing a new type of options.

Definition: A **Canadian option** is an option which expires after an exponentially distributed random time τ, independent of the asset process.

We have introduced thus three type of options, according to whether the expiration is:

1. At a fixed expiration time t, at which a **final payoff** $h_f(X_t)$ is earned; these are called **European options**.
2. At the first time T_l when the process X_t crosses some barrier l, at which time a **rebate** $h_b(X_{T_l})$ is received; these will be called pure **Barrier options**.
3. At an exponentially distributed random time τ, independent of the asset process; these will be called **Canadian options**.

While alive, each type of option may also earn a continuous "interest" $c(X_t)$.

scr **Review of Section 3:** We have learned how to valuate pure Barrier options via differential equations. Provided we assume as usual an exponential Brownian motion model $S_t = S_0 e^{X_t}$ where X_t is Brownian motion with drift and provided we express the payments $h_b(x), c(x)$ in terms of the variable $X_t = \log(S_t/S_0)$, their value depends only on the initial starting point x

$$v(x) = \mathbb{E}^*_x e^{-r\,t} h_b(X_{T_l}) + \int_0^{T_l} e^{-rt} c(X_s) ds$$

satisfies the differential equation:

$$\frac{\sigma^2}{2} v''(x) + \mu v'(x) - r v(x) + c(x) = 0 \tag{35}$$

$$v(l) = h_b(l) \tag{36}$$

where $\mu = r - \frac{\sigma^2}{2}$ (to ensure risk neutrality of the valuation measure). Thus, the boundary payment and the continuous "interest" enter the differential equation as boundary conditions and nonhomogeneous terms, respectively.

We have not discussed yet the differential equation approach for the first and most common options, the European options, which were valued by a different approach, making use of the explicit formula available for the density of exponential Brownian motion assets at any finite time. The reason is that the value of European options with finite time expiration depend not only on the initial starting point x, but also on the remaining time until expiration t. As such, they satisfy not ordinary, but **partial differential equations** (PDE's). The purpose of this section is to show how Canadian options, whose value is also given by ordinary differential equations, may be used to approximate the more complicated European options.

Even in the simplest case of no barrier payoff and no interest

$$v(t, x) = \mathbb{E}^*_x e^{-r\,t} h_f(X_t)$$

we are lead already to the PDE

$$Gv - rv - \frac{\partial}{\partial t}v = 0 \tag{37}$$
$$v(0,x) = h_f(x) \tag{38}$$

where $Gv(x) = \frac{\sigma^2}{2}v''(x) + (r - \frac{\sigma^2}{2})v'(x)$

In the general case of European options which have also a barrier payoff and continuous interest, we are lead to:

$$Gv - rv + c(x) - \frac{\partial}{\partial t}v = 0 \tag{39}$$
$$v(0,x) = h_f(x) \tag{40}$$
$$v(t,l) = h_b(l) \tag{41}$$

These PDE's may be derived by the usual method of conditioning on one step and employing the random walk approximation, but we skip the derivations. Solving partial diffential equations is very rarely possible analytically; they are typically solved numerically using computers, which are not featured in our notes.

However, the Canadian options last introduced can provide quite good approximations for the value of European options. The idea is to approximate a European option with remaining lifetime t via a Canadian option whose exponential lifetime τ has the rate λ chosen by $\lambda = \frac{1}{t}$, so that its expected duration is precisely $\mathbb{E}\tau = t$.

We will show next that the value of Canadian options is independent of time and may be obtained by solving ordinary differential equations, instead of partial ones. Probabilistically, the independence of time is a consequence of the memoryless property of the exponential lifetime. Analytically, it will be seen from the close relation between Canadian options and Laplace transforms, discussed in the next section.

Notes: 1) Perpetual options are a particular case of Canadian options when $\lambda = 0$.

2) The independence of time makes Canadian options simpler to price than European options (just like perpetuals and pure barrier options). However, Canadian options are not traded, and are not likely to ever be. Their practical implementation would require that the seller maintains some type of "exponential clepsydra" and that the buyer trusts the seller for letting him know when the option expired).

8.1 The Connection with Laplace transforms

Let $v(t,x) = \mathbb{E}^* e^{-r\,t} h_f(X_t)$ denote the value of an European option with fixed expiration t and some final payoff $h_f(x)$. Then,

Lemma 8.1. *The randomized or "Canadian" value of the option, defined as* $v^*(\lambda, x) = \mathbb{E}^*_x e^{-r\tau} h_f(X_\tau)$ *where* τ *is exponential with rate* λ *is given by:*

$$v^*(\lambda, x) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} v(t, x) dt$$

where $\lambda = 1/t$.

Proof: Conditioning on $\tau = t$ we find:

$$\begin{aligned} v^*(\lambda, x) &= \mathbb{E}^*_x \int_{t=0}^{\infty} \lambda e^{-\lambda t} dt \left(e^{-rt} h_f(X_t)\right) \\ &= \int_{t=0}^{\infty} \lambda e^{-\lambda t} dt \left(\mathbb{E}^*_x e^{-rt} h_f(X_t)\right) = \int_{t=0}^{\infty} \lambda e^{-\lambda t} v(t, x) dt \end{aligned}$$

Note: By the lemma above, we see that the Canadian value differs from the Laplace transform of the European value only by the extra factor λ. This "almost" Laplace transform, in which we average with respect to the exponential density, is called Laplace-Carson transform.

One of the most important feature of Laplace transforms $\hat{f}(\lambda) = \int_{t=0}^{\infty} e^{-\lambda t} f(t) dt$ is that they transform derivative operations in multiplications, via the well known:

Lemma 8.2. *The Laplace transform of a derivative is given by:*

$$\hat{f'} = \lambda \hat{f} - f(0)$$

By this property, taking Laplace transforms of differential equations changes them into algebraic equations. The Laplace-Carson transform $f^*(\lambda) = \int_0^i e^{-\lambda t} f(t) dt$ has of course the same properties as the Laplace transform (plus the extra convenience of transforming constants into constants), which is why considering Canadian values will rid us of the undesired partial with respect to time.

The formula for transforming derivatives becomes:

Lemma 8.3. *The Laplace-Carson transform of a derivative is given by:*

$$(f')^*(\lambda) = \lambda(f^* - f_0).$$

Applying now a Laplace-Carson transform in time, the partial differential equation (38) turns into an ordinary differential equation.

Lemma 8.4. *The Canadian value of a derivative with final payoff* h_f

$$v^*(\lambda, x) = \mathbb{E}^*_x e^{-r\tau} h_f(X_\tau)$$

satisfies the ordinary differential equation:

$$Gv^* - rv^* - \lambda(v^* - h_f) = Gv^* - (r + \lambda)v^* + \lambda h_f = 0$$

The proof of this lemma is left as an exercise.

Note: The differential equation above is precisely the same as that obtained for a perpetual option yielding continuous dividend $\lambda h_f(X_s)$, if the interest rate was $r+\lambda$. Thus,

$$v^* = \mathbb{E}_x \int_0^\infty e^{-(r+\lambda)s} \lambda h_f(X_s) ds$$

Example 1: Discontinuous payoffs Find the value of the **Canadian digital**

$$v(S) = \mathbb{E}_S e^{-r\tau} I_{S_\tau \geq K}$$

Solution: Converting to the variables $X_t = \log(S_t/S_0)$ we find the **converted** final payoff

$$I_{S_\tau \geq K} = I_{S_0 e^{X_t} \geq K} = I_{X_\tau \geq k}$$

where $k = \log(K/S_0)$. By lemma 5.4, $v(x)$ must satisfy :

$$\frac{\sigma^2}{2} v''(x) + (r - \frac{\sigma^2}{2}) v'(x) - (r+\lambda) v + \lambda I_{x \geq k} = 0$$
$$v(\pm\infty) \quad \text{not exponentially increasing}$$

One way to solve the differential equation above is by taking one more Laplace transform, which will change all the derivatives into multiplication operations. However, a direct approach is also possible.

Because the payoff is given by different formulas below and above k, we must solve each part separately. Let θ_1, θ_2 denote respectively the positive and negative roots of the characteristic equation $\frac{\sigma^2}{2}\theta^2 + (r - \frac{\sigma^2}{2})\theta - (r+\lambda)$ of the homogeneous operator.

Below k we find $v(x) = A_1 e^{\theta_1(x-k)}$ and above k we find $\frac{\lambda}{r+\lambda} + A_2 e^{\theta_2(x-k)}$. To complete the solution, we need two more boundary conditions! Intuitively, the boundary value for each piece should be provided by the way it fits to the other piece. This leads to the following recipe:

Smooth fit recipe: The boundary conditions necessary for fitting several solutions of an ODE on different subdomains are provided by assuming "smooth fit", i.e by equating values of the function and as many of its derivatives as necessary on both sides.

In this case, we find:

$$A_1 = \frac{\lambda}{r+\lambda} + A_2$$
$$A_1\theta_1 = A_2\theta_2$$

which yields $A_1 = \frac{\lambda}{r+\lambda}\frac{\theta_2}{\theta_2-\theta_1}$, $A_2 = \frac{\lambda}{r+\lambda}\frac{\theta_1}{\theta_2-\theta_1}$. In terms of the original variables, the solutions is:

$$v(x) = \begin{cases} \frac{\lambda}{r+\lambda}\frac{\theta_2}{\theta_2-\theta_1}(S/K)^{\theta_1}) \text{ if } S > K \\ \frac{r}{\lambda+r}(1 + \frac{\theta_1}{\theta_2-\theta_1}(S/K)^{\theta_2}) \text{ if } S \geq K \end{cases}$$

Example 2: The Canadian put and put-call parity

a) Show that the value of the Canadian put $v(S) = \mathbb{E}_S e^{-r\tau}(K - S_\tau)_+$ above the strike price K is given by $p(S) = \frac{K}{\theta_1-\theta_2}\left(1-\theta_1\frac{r}{r+\lambda}\right)(S/K)^{\theta_2}$ and the value below the strike price is given by $K\frac{\lambda}{r+\lambda} - S + c(S)$ where $c(S) = \frac{K}{\theta_1-\theta_2}\left(1-\theta_2\frac{r}{r+\lambda}\right)(S/K)^{\theta_1}$. The same function $c(S)$ is shown in Exercise 5.5 to provide the value of the Canadian call below the exercise price.

b) Conclude that the Canadian call and Canadian put verify the put-call parity formula $C = F - \hat{K} + P$ where $F, \hat{K}$ denote the forward value of the asset and of the exercise price.

Solution

The value of the **Canadian put** in terms of the exponent variable x must satisfy:

$$\begin{aligned}
&\frac{\sigma^2}{2}v''(x) + (r - \frac{\sigma^2}{2})v'(x) - (r+\lambda)v + \lambda(K - S_0e^x)_+ = 0\\
&v(\infty) = 0\\
&v(-\infty) \quad \text{not exponentially increasing}
\end{aligned}$$

Above $x = k$ this yields $v(x) = A_2e^{\theta_2(x-k)}$ and below k it yields $v(x) = K\frac{\lambda}{r+\lambda} - S_0e^x + A_1e^{\theta_1(x-k)}$.

The smooth fit recipe applied at k yields:

$$\begin{aligned}
A_2 &= -K\frac{r}{r+\lambda} + A_1\\
A_2\theta_2 &= A_1\theta_1 - K
\end{aligned}$$

In terms of the original variables, this leads to:

$$v(S) = \begin{cases} \frac{K}{\theta_1-\theta_2}\left(1-\theta_1\frac{r}{r+\lambda}\right)(S/K)^{\theta_2} & \text{if } S > K\\ K\frac{l}{\lambda+r} - S + \frac{K}{\theta_1-\theta_2}\left(1-\theta_2\frac{r}{r+\lambda}\right)(S/K)^{\theta_1} & \text{if } S \geq K \end{cases}$$

Thus, above K we have $C(S) = S - \hat{K} + p(S)$ and below K we have $P(S) = \hat{K} - S + c(S)$, and so put call parity holds in both cases.

We also consider **Canadian barrier options** which may either expire either at the first time T_l when the process X_t crosses a barrier l, or after an exponentially distributed random time τ with expectation $\mathbb{E}\tau = t$. The expiration time is thus $T = \min(T_L, \tau)$. The option may earn either a final payoff $h_f(X_\tau)$ or a **rebate** $h_b(X_{T_l})$ depending on how expiration occurred. Also, a continuous interest $c(X_t)$ may be earned while the option is alive.

Lemma 8.5. *The value of a Canadian barrier option with expiration time $T = \min(T_L, \tau)$, different boundary and expiration payoffs h_b, h_f and dividend $c(x)$*

$$v^*(x) = \mathbb{E}_x^*\left(e^{-rT}h_f(X_T)I_{\tau\leq T_l} + e^{-rT}h_b(X_T)I_{T_l\leq\tau} + \int_0^T e^{-rs}c(X_s)ds\right)$$

satisfies the differential equation:

$$\begin{aligned} Gv^* - (r+\lambda)v^* + \lambda h_f + c = 0 \\ v^*(l) = h_b(l) \end{aligned}$$

The proof is similar to that of Lemma 5.4.

We conclude with a discussion of very popular type of options:

8.2 American options **

American options are options which confer to their buyer the right of exercise at any moment preceding their expiration time t. The payoff of an American put option for example is $e^{r\,T_t}\,(K - S_{T_t})_+$, where $T_t = min(T_L, t)$, and T_L denotes the hitting time of an exercise boundary $L_s, 0 \le s \le t$, to be chosen by the customer. By risk neutral valuation, the value of this option is given by

$$v(S,t) = \max_L \mathbb{E}^*_S\, e^{r\,T_t}\,(K - S_{T_t})_+$$

where $\mathbb{E}^*$ denotes a risk neutral measure. This problem is difficult even numerically, due to the fact that the optimal exercise boundary L depends on time. More precisely, the optimal exercise point at time s is some function $L(t-s)$ of the remaining time until expiration.

The problem's difficulty arose the interest in analytic approximations. Already in 1965, Samuelson and H. McKean [?], proposed in what was maybe the first paper in mathematical finance to approximate the problem by that of pricing a "perpetual" option (with infinite expiration time $t = \infty$). For the perpetual, the dependence on the remaining time disappears, leaving us with the problem of valuation of a barrier put with fixed barrier L, followed by the optimization of L. The idea of considering constant exercise barriers, while maintaining a finite expiration time is what lead Carr to introducing the Canadian options. We will start by pricing a product called "capped American option" in which the constant boundary L is not chosen, but imposed, and in addition the expiration time is exponentially distributed.

The value of a "Canadian capped American put" is thus

$$b_S = \mathbb{E}\, e^{-rT}\,(K - S_T)_+$$

where $T = \min(\tau, T_L)$.

In terms of the exponent variable x the value of the "Canadian capped American put" must satisfy:

$$\begin{aligned} &\frac{\sigma^2}{2}b''(x) + (r - \frac{\sigma^2}{2})b'(x) - (r+\lambda)b(x) + \lambda(K - S_0 e^x)_+ = 0 \\ &b(\infty) = 0 \\ &b(l) = K - L \end{aligned}$$

This could be solved by splitting the state space in two pieces above and below $k = \log(K/S_0)$. We present however a different idea of splitting the value of this option in two parts $b(x) = p(x) + a(x)$, where $p(x)$ is the value of the European put and the difference $a(x)$ represents the **early exercise premium**.

The advantage is that the early exercise premium is a "pure barrier" option (with 0 final payoff), i.e it satisfies the somewhat easier problem:

$$\begin{aligned}
&\frac{\sigma^2}{2}a''(x) + (r - \frac{\sigma^2}{2})a'(x) - (r+\lambda)a(x) = 0\\
&a(\infty) = 0\\
&a(l) = (K - L) - p(l)
\end{aligned}$$

The solution is $a(S) = K\frac{r}{r+\lambda}(1 - \frac{\theta_1-\theta_2}{1-\theta_2}(L/K)^{\theta_1}(S/L)^{\theta_2}$.

Note: The decomposition $b(x,t) = p(x,t) + a(x,t)$ holds also for usual American options and nonconstant exercise boundaries; the equation for the early exercise premium becomes then:

$$\begin{aligned}
&\frac{\sigma^2}{2}a''(t,x) + (r - \frac{\sigma^2}{2})a'(t,x) - (r+\lambda)a(t,x) = 0\\
&a(t,x) = 0\\
&a(s,\infty) = 0, \forall 0 \le s \le t\\
&a(t,l) = (K - L) - p(t,l)
\end{aligned}$$

The last boundary condition has the clear interpretation that the holder of an American option who has already received the European payoff at the start should receive if he exercises only the difference between the boundary payoff and the current value of the european payoff.

Exercise ** a) Compute the value of the early exercise premium for a Canadian put capped at a boundary L.

b) By optimizing L, find the optima exercise barrier and the value of the Canadian American put.

Ans: Optimal choice: $\tau = \tau_b$ with b determined by $\frac{\partial}{\partial b}v = 0$ (or, equivalently, by $\frac{\partial}{\partial x}v = \frac{\partial}{\partial x}(K - e^x)$ called "smooth fit"). (Samuelson-McKean, 1965)

8.3 Exercises

Exercise 8.1 Prove lemmas 5.2, 5.3. **Hint:** Use the definition of the Laplace transform and integration by parts.

Exercise 8.2 Prove lemma 5.4 **Hint:** Apply the Laplace-Carson transform to the differential equation (38).

Write down the differential equations satisfied by the values of the following options (as functions of the initial starting point $x = \log(S/S_0)$), and finally express them as functions of the asset current price S.

Exercise 8.3 The Canadian zero coupon bond $v(S) = \mathbb{E}_S e^{-r\tau}$ **Ans:** $v(S) = \frac{\lambda}{\lambda+r}$.

Exercise 8.4 The Canadian asset or nothing $v(S) = \mathbb{E}_S e^{-r\tau} S_\tau I_{S_\tau \geq K}$ **Ans:**

$$v(S) = \begin{cases} K\frac{1-\theta_2}{\theta_1-\theta_2}(S/K)^{\theta_1} & \text{if } S < K \\ K\frac{1-\theta_1}{\theta_1-\theta_2}(S/K)^{\theta_2} + S & \text{if } S > K \end{cases}$$

Exercise 8.5 The Canadian call option

$$v(S) = \mathbb{E}_S e^{-r\tau}(S_\tau - K)_+ I_{S_\tau \geq K}$$

Ans:

$$v(S) = \begin{cases} S - K\frac{\lambda}{r+\lambda} + \frac{K}{\theta_1-\theta_2}\left(1 - \theta_1\frac{r}{r+\lambda}\right)(S/K)^{\theta_2} & \text{if } S > K \\ \frac{K}{\theta_1-\theta_2}\left(1 - \theta_2\frac{r}{r+\lambda}\right)(S/K)^{\theta_1} & \text{if } S < K \end{cases}$$

Exercise 8.6 Canadian "down and in" zero-coupon bond

$$v(S) = \mathbb{E}_S e^{-r\,T_L} I_{T_L \leq \tau}$$

Exercise 8.7 Canadian "down and out" zero-coupon bond

$$v(S) = \mathbb{E}_S e^{-r\,T_L} I_{\tau \leq T_L}$$

Exercise 8.8 Canadian continuous interest below barrier from one currency unit:

$$v(S) = \mathbb{E}_S \int_0^T e^{-rt} r I_{\{S_t \leq L\}} dt$$

8.4 Solutions

Solution 8.1

Solution 8.2

Solution 8.3 The value of the **Canadian zero coupon bond**

$$v(x) = \mathbb{E}_x e^{-r\tau}$$

must satisfy the system

$$\begin{aligned} \frac{\sigma^2}{2} \quad & \frac{\partial^2}{\partial x^2} v + (r - \frac{\sigma^2}{2}) \frac{\partial}{\partial x} v - (r+\lambda) v + \lambda = 0 \\ & v(-\infty), v(\infty) \quad \text{not exponentially increasing} \end{aligned}$$

which yields $v(x) = \frac{\lambda}{\lambda + r}$.

Solution 8.4 The payoff of the **Canadian asset or nothing** in terms of $X_t = \log(S_t/S_0)$ is $S_0 e^X_t I_{X_t \geq k}$, where $k = \log(K/S_0)$.

$$v(x) = \mathbb{E}_x e^{-r\tau} S_0 e^X_\tau I_{X_\tau \geq k}$$

must satisfy the system

$$\begin{aligned} \frac{\sigma^2}{2} \quad & \frac{\partial^2}{\partial x^2} v + (r - \frac{\sigma^2}{2}) \frac{\partial}{\partial x} v - (r+\lambda) v + \lambda S_0 e^x I_{x \geq k} = 0 \\ & v(-\infty), v(\infty) \quad \text{not exponentially increasing} \end{aligned}$$

When $x \leq k$ we must have $v(x) = A_1 e^{\theta_1 (x-k)}$ and above k we have $v(x) = A_2 e^{\theta_2 (x-k)} + S_0 e^x$, where $\frac{\sigma^2}{2}\theta_i^2 + (r - \frac{\sigma^2}{2})\theta_i - (r+\lambda) = 0, \theta_1 < 0,\ \theta_1 > 0, \theta_2 < 0$.

The smooth fit conditions at k yield $A_1 = A_2 + K, \theta_1 A_1 = \theta_2 A_2 + K,\ A_2 = K\frac{1-\theta_1}{\theta_1 - \theta_2}, A_1 = K\frac{1-\theta_2}{\theta_1-\theta_2}$ which yields

$$v(S) = \begin{cases} K\frac{1-\theta_2}{\theta_1-\theta_2}(S/K)^{\theta_1} \text{ if } S < K \\ K\frac{1-\theta_1}{\theta_1-\theta_2}(S/K)^{\theta_2} + S \text{ if } S > K \end{cases}$$

Solution 8.5

The value of the **Canadian call option** may be optained as $C = A - KB$, where A denotes the value of the asset or nothing option, and B that of a digital, yielding:

$$v(S) = \begin{cases} S - K\frac{\lambda}{r+\lambda} + \frac{K}{\theta_1-\theta_2}\left(1 - \theta_1 \frac{r}{r+\lambda}\right)(S/K)^{\theta_2} \text{ if } S > K \\ \frac{K}{\theta_1-\theta_2}\left(1 - \theta_2\frac{r}{r+\lambda}\right)(S/K)^{\theta_1} \qquad \text{if } S < K \end{cases}$$

Solution 8.6 The value of the **Canadian "down and in" barrier "zero coupon bond"** is

$$v(x) = \mathbb{E}_x e^{-r\,T} I_{T_l \leq \tau}$$

where $l = \log(L/S_0)$ is the fixed barrier for the exponent and T_l is the first hitting time of l by the exponent $X_t = Log(S_t/S_0)$. It must satisfy the system

$$\begin{aligned}&\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}v + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial x}v - (r+\lambda)v = 0\\ &v(l) = 1\\ &v(\infty) \quad \text{not exponentially increasing}\end{aligned}$$

The solution is: $v(x) = e^{\theta_2(x-l)}, v(S) = (S/L)^{\theta_2}$.

Solution 8.7 Canadian "down and out" barrier "zero coupon bond"

$$v(x) = \mathbb{E}_x e^{-r\,T} I_{\tau \leq T_l}$$

must satisfy the system

$$\begin{aligned}&\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}v + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial x}v - (r+\lambda)v + \lambda = 0\\ &v(l) = 0\\ &v(\infty) \quad \text{not exponentially increasing}\end{aligned}$$

The solution is: $v(S) = \frac{\lambda}{\lambda+r}\left(1 - (S/L)^{\theta_2}\right)$.

Solution 8.8 Canadian continuous interest below barrier option

$$v(x) = \mathbb{E}_x \int_0^T e^{-rs} r I_{\{X_s \leq l\}} ds$$

Applying the general recipe with $h_f = h_b = 0, c(x) = I_{\{x \leq l\}}$, we get the system

$$\begin{aligned}&\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}v + (r - \frac{\sigma^2}{2})\frac{\partial}{\partial x}v - (r+\lambda)v + rI_{x\leq l} = 0\\ &v(l) = 0\\ &v(-\infty), v(\infty) \quad \text{not exponentially increasing}\end{aligned}$$

which yields $v(x) = \begin{cases} a_2 e^{\theta_2(x-l)}) \text{ if } x > l \\ \frac{r}{\lambda+r} + a_1 e^{\theta_1(x-l)}) \text{ if } x \leq l \end{cases}$

The smooth fit at b yields $a_1 = \frac{r}{\lambda+r}\frac{\theta_2}{\theta_1-\theta_2}, a_2 = \frac{r}{\lambda+r}\frac{\theta_1}{\theta_1-\theta_2}$

PART II

1 Martingales

A sequence of random variables S_i has the **martingale property** if the conditional expectation of its increments based on all the past " information" F_t available up to time t is 0.

$$\mathbb{E}[(S_{t+1} - S_t)/F_t] = 0$$

Typically, the " information" F_t available up to time t is the values of all the previous values of the sequence i.e. $F_t = \{S_1, S_2, ..., S_t\}$. Equivalently, the martingale property means the conditional expectation of the next coming value S_{t+1} equals precisely the last observed value S_t

$$\mathbb{E}[S_{t+1}/F_t] = S_t \tag{42}$$

This just means that the distribution of the next value S_{t+1} is centered around the previously observed value S_t.

We have already seen some simple examples of martingales.

Example 1.1 (**Addititive martingales**) It is easy to see that an additive asset process is a martingale iff the increments have mean 0, since the conditional expectation of its increments equals the unconditional mean, for any value of the " information" F_t.

Or, taking conditional expectations on both sides of the additive formula $S_{t+1} = S_t + X_{t+1}$, where X_t denotes the increment at time t, we have:

$$\mathbb{E}[S_{t+1}/F_t] = \mathbb{E}[S_t + X_{t+1}/F_t] = S_t + \mathbb{E}X_{t+1} = S_t$$

Additive martingales are used to model gambling.

Example 1.2 (**Multiplicative martingales**) It is easy to see that a multiplicative asset process is a martingale iff the factors have mean 1. Indeed, taking conditional expectations on both sides of the recursive additive formula $S_{t+1} = S_t X_{t+1}$ we get:

$$\mathbb{E}[S_{t+1}/F_t] = \mathbb{E}[S_t Y_{t+1}/F_t] = S_t \mathbb{E}Y_{t+1} = S_t$$

Multiplicative martingales are used to model the evolution of financial instruments like stocks.

Example 1.3 Find a so that the multiplicative process a^{Y_n} is a martingale, where $Y_n = \sum_{i=1}^{n} Z_i$ is a biased random walk with probabilities of moving to the right(left) p (q), i.e. $Z_i = \pm 1$.

Solution Note that indeed $a_n^Y = \prod_{i=1}^{n} (a)^{Z_i}$ is a multiplicative process. It only remains to find a so that the factors have expectation 1. Since Z_i equals 1 with probability p and -1 with probability q, a must satisfy: $\mathbb{E}[a^{Z_i}] = pa + qa^{-1} = 1$, which yields $a = 1$ (not interesting) and $a = fracqp$.

We will give now an example of a more complicated martingale.

Example 1.4 The Wright Fischer model

This example models the evolution of a gene's frequency on an "island" which may support only a fixed population of N individuals. The assumption is that X_{n+1}, the number of this gene's carriers at time $n+1$ has the distribution Bin_{N,p_n} where p_n is the frequency of carriers in the n'th generation $p_n = \frac{X_n}{N}$.

Note that in this example, the distribution of the increment $X_{n+1} - X_n$ depends on X_n, so this is **not a random walk**. Also, treating this example by the method of conditioning after one step would involve difference equations involving N terms!

However, this complicated process is still a martingale, since

$$\mathbb{E}[X_{n+1}/X_n] = Np_n = X_n$$

Note: If X_n ever hits either 0 or N it becomes absorbed there. Can you guess what will happen on a "Wright Fischer island" after 100 billions of zillions of generations?

Example 1.5 (**The extended martingale property**) Show that a martingale sequence satisfies also:

$$\mathbb{E}[X_{t+k}/F_t] = X_t \quad \text{for any } k \geq 1$$

Note: It is quite easy to verify statements like the above for additive and multiplicative martingales, since conditional expectations reduce immediately to unconditional ones. This easy case is assigned to the reader (Exercise 1). To extend this to the much wider class of general martingales, one needs to use in addition the **law of conditional expectations**

$$\mathbb{E}[\mathbb{E}[X/Y]] = \mathbb{E}[X] \tag{43}$$

as illustrated below. Since this law is more advanced conceptually, we may take occasionally the short cut of giving proofs for the simpler case of additive or multiplicative martingales.

Solution The proof, very similar to that of Exercise 1, consists in writing

$$X_{t+k} = X_t \sum_{i=1}^{k} Z_{t+i}$$

and in taking conditional expectations:

$$E[X_{t+k}/F_t] = E[X_t + \sum_{i=1}^{k} Z_{t+i}/F_t]$$

and we are left with proving that $E[Z_{t+i}/F_t] = 0$ for any $i \geq 1$. This may be done by induction. It is true for $i = 1$, so suppose we proved it up to $i = j$. To get the result for $j+1$ we condition on the information available at time j, using the law of conditional epectations $E[Z_{t+j+1}/F_t] = E\ E[Z_{t+j+1}/X_1, \ldots, X_t, X_{t+1}, \ldots, X_{t+j} = E[E[Z_{t+j+1}/F_{t+j}]] = E[0] = 0$.

We describe next the original introduction of martingales in the context of gambling.

1.1 Martingales in gambling

We consider an additive martingale $S(t) = \sum_{i=1}^{t} Z_i$, where $Z_i = u_i, l_i$ with probabilities p_i, q_i. s_i represent the quantities gambled at the i'th step and S_t represents the cumulative gambler's wealth at time t. It is assumed that the game is "fair", i.e.

$$\mathbb{E}Z_i = p_i u_i + q_i l_i = 0$$

The gambler is thus allowed to chose the stakes u_i, l_i, the odds p_i, q_i (subject to the "fairness" constraint, and, most importantly, a **stopping strategy** T. An example is $T = min(T_L, T_K$, i.e. stopping either the first time when the wealth reaches or overshoots a prescribed target K or when a prespecified level of losses of at least L is incurred. Letting T_a denote the first time of overshooting a point a, the game's duration (or exit time from $[L, K]$) will be thus $T = min(T_L, T_K)$.

The gambler's purpose is to optimize his expected winnings by chosing the stopping bounds (L, K), as well as the values and the odds chosen at time i.

A technical assumption: We may allow in general u_i, l_i, p_i to depend on the whole "history" of the game up to time $i-1$, F_{i-1}, i.e. allow them to be some functions of $\{S_1, S_2, ..., S_{i-1}\}$, but we may not allow them to be functions of future values $S_i, S_{i+1}, ...$, since this would mean that we allow in our framework "preknowledge" of the future.

The most useful result of martingale theory, the optional stopping theorem, states that no matter how clever the gambler tries to be, subject to some reasonable restictions stated below, the gambler cannot escape the law

$$E[S_T] = S_0$$

Thus, it is impossible to improve on the average on your initial capital, at least if you are forced to obey some sensible restrictions specified below (like gambling for a time T with finite expectation and keeping the amount of your losses bounded).

1.2 The optional stopping theorem

Theorem 1.1 **(The optional stopping theorem)** If S_t is a martingale and T is a random stopping time, then

$$\mathbb{E}S_T = S_0$$

if any of the following conditions hold:

(1) a) $T < \infty$ a.s. and b) $\max_{\{1 \le t \le T\}} |S_t|$ is bounded by a constant C.

Informally, this means the game is sure to end and the capital remains always bounded.

(2) a) $\mathbb{E}T < \infty$ and b) $\max_{\{1 \le t \le T\}} |S_t - S_{t-1}|$ is bounded by a constant C.

In this case, we require the stronger condition on termination of finite expected termination (thus, the probabilities of this going on a long time decrease reasonably fast), and the weaker condition that the stakes are bounded (but the capital need not be bounded).

(3) T is bounded.

With such a strong assumption on T, no assumption is needed on S_t.

Notes: 1) This result is a generalization of the obvious fact that if S_t is a sum of increments with 0 mean then $\mathbb{E}S_t = S_0$ for any fixed t. The fact that we may extend this to the case of arbitrary stopping times T has the interpretation that even the most clever stopping rules T (which obey the restrictions above) cannot break the odds.

2) While the assumptions of the optional stopping theorem may look at first technical, they have however a clear meaning: by using "reckless" strategies (with unbounded stakes or borrowing) for very long times, a gambler may "beat" the odds. This will be illustrated in the example of the doubling the bets strategy, originally called "martingale", which gave this field its name.

Example 1.6 Expected win in Gambler's ruin problem

The Gambler's ruin problem is the case $u_i = -\lambda_i = 1, p_i = q_i = \frac{1}{2}$.

A direct application of the optional stopping theorem to the the martingale X_t representing the total capital yields

$$v(x) = \mathbb{E}_x X_T = x$$

This is valid by case 1) of the optional stoping theorem. Indeed, cleary $|X_t| \le \max(|L|, K)$. Also, as known, the finite state Markov chain X_t must be positively recurrent, and so the probability that it never reaches either of the boundaries of the finite interval $[L, K]$ is 0 and thus $T < \infty$ a.s.

Note: This problem was also solved in Exercise 2.1.b) by the method of difference equations.

Example 1.7 Expected frequency at absorption time on "Wright-Fischer's island"

We stop observing "Wright-Fischer's island" at the time $T = \min(T_0, T_N)$. What is the expected frequency of the special gene?

Example 1.8 The probabilities of escape

Consider again the martingale X_t from a gambler's ruin problem, or that of the count of the special gene on "Wright-Fischer's island" observed until $T = \min(T_L, T_K)$ (here, we are mainly interested in $L = 0, K = N$). Find the probability $p_x = \ Pr_x\{X(T) = K\}$ (say, that a gambler starting with capital x will end up rich, as opposed to bankrupt).

As observed above, by the optional stopping theorem applied to the martingale X_t

$$\mathbb{E}_x X_T = \mathbb{E}_x X_0 = x$$

On the other hand by conditioning on the final outcome we have:

$$\mathbb{E}_x X_T = K\ \mathbb{P}_x\{X_T = K\} + L\ (1 - \mathbb{P}_x\{X_T = K\}) = x,$$

which gives

$$p_x = \mathbb{P}_x\{X_T = K\} = \frac{(x - L)}{(K - L)}.$$

A simpler proof of this was found in Exercise 2.1.a) by solving the difference equations satisfied by p_x. That method could not solve however the Wright Fischer's model. Form our new vantage point, we see however that the two results are identical!

Example 1.9 Find the probability $p_x = P_x\{S_{min[T_L, T_K} = K\}$ for exponential martingale Brownian motion (i.e. with $g = \frac{-\sigma^2}{2}$). *Hint: the answer is the same for any martingale!*

Example 1.10 The doubling "martingale" strategy

We examine now the strategy which gave martingales their names (nowadays outlawed in casinos).

A gambler with no initial capital has as goal to win 1 pound. His first bet is $s_1 = 1$ pound. If he loses, he bets whatever it takes to bring him up to 1 pound ($s_2 = 2$ pounds at the second bet, $s_3 = 4$ at the third, and in general $s_n = 2^{n-1}$ on the n'th bet. The stopping time is T_1. We note immediately that this strategy creates a dollar out of nothing and does not satisfy the optional stopping theorem, i.e.

$$E_0 X_{T_1} = 1 > 0!!$$

We examine now the conditions of the optional stopping theorem. It is easy to check that $p_k = \mathbb{P}\{T = k\} = 2^(-k), k = 1, 2, ...$ and thus both condition 1 a) (that $\sum_k p_k = 1$) and condition 2 a) (that $\mathbb{E}T = \sum_k k\ p_k = 2 < \infty$) are satsified. However, neither the cumulative fortune, nor the stakes are bounded, since the loss may double its value an arbitrary number of times and of course the gambling time does not have to be bounded. Thus, neither condition 1 b) nor 2 b) are satisfied.

Notice that this strategy seems quite efficient for the gambler (a sure win in a number of steps with expectation 2!). Also, practically, it seems at first safe for the bank too, since in practice the gamblers will have to limit the time they gamble by some finite number n, and then the optional stopping theorem will apply (by any of the three conditions!). Note that the possible loss after the n'th bet is $-2^n + 1$. The 0 expectation of the optional stopping theorem means in practice roughly that the winnings of 2^n successful martingale gamblers will be outset by the huge loss of one misfortunate; the fear that this loss will not be honoured is what lead to the outlawing of this strategy.

More precisely, if all martingale gamblers bound their losses at $L = -2^n + 1$, then we are alowed to apply the optional stopping theorem, and find as usual that the fraction of winning martingale gamblers $p_0 = \frac{L}{1+L} = \frac{2^n-1}{2^n}$ is very close to 1. The fraction of losers $1 - p_0 = 2^{-n}$ is very small, but the potential loss is huge $2^n - 1$, averaging thus to 0. When $L \to \infty$ the bad second case somehow disappears by indefinite postponement)!

Note: The expected duration may also be found to be $t_0 = E_0 T = 2 - 2^{-n}$ by setting up a corresponding difference equation, for example.

1.3 Wald's martingale **

One of the most useful results of martingale theory is that for any Levy process X_t with cumulant function $c(\theta)$ (which is given by the equation $\mathbb{E}e^{\theta X_t} = e^{tc(\theta)}$), the process:

$$M_t = e^{\theta X_t - \delta t} - (c(\theta) - \delta) \int_0^t e^{\theta X_s - \delta s} ds$$

is a martingale.

Exercise 1.1** Prove that M_t is a martingale.

Let now τ denote the exit time from a certain interval $I = [x_1, x_2]$. Under either of the usual conditions on the stopping time, we can conclude that:

$$\mathbb{E}_x e^{\theta_\delta X_\tau - \delta\tau} = e^{\theta_\delta x} \tag{44}$$

provided that $\theta = \theta_\delta$ is a root of the "Wald" equation $c(\theta) = \delta$ (so that the integral term does not appear).

We show now that this equation determines in principle the distribution of the hitting time.

In the case X_t is Brownian motion with variance 1 and drift μ (generator $Gf = \frac{f''}{2} + \mu f'$ and cumulant function $c(\theta) = \frac{\theta^2}{2} + \mu\theta$), the problem simplifies considerably since the continuity of Brownian motion implies that at the exit time X_τ must equal either x_1 or x_2 and thus the only unknown remaining in (44) is the exit time. We also note that $c(\theta)$ is a convex function. This implies that Wald's equation always has two roots, if δ is larger than the minimum of the quadratic.

We compute now the expectation $f(x) = \mathbb{E}_x e^{-\delta\tau}$ by breaking it in two cases, denoted by $f_i(x) = \mathbb{E}_x e^{-\delta\tau} I_{\{X_\tau = x_i\}}, i = 1, 2$.

For each of the roots θ_j of Wald's equation, we find:

$$\mathbb{E}_x e^{\theta_j X_\tau - \delta\tau} = \sum_i \mathbb{E}_x e^{\theta_j X_\tau - \delta\tau} I_{\{X_\tau = x_i\}} = e^{\theta_j x_1} f_1(x) + e^{\theta x_2} f_2(x) = e^{\theta_j x}$$

This provides two equations for f_1, f_2 (one for each root of Wald's equation). Adding the two solutions, we find:

$$f(x) = f_1(x) + f_2(x) = a_1 e^{\theta_1 x} + a_2 e^{\theta_2 x}$$

where $a_1, a_2 = ...$

Note that in this case f_x may be obtained more easily as a solution of the (Feynman-Kac differential equation):

$$\begin{aligned} \frac{f''}{2} + \mu f' - \delta f &= 0 \\ f(x_1) &= 1 \\ f(x_2) &= 1 \end{aligned}$$

In the general jump-diffusion case with X_t being the sum of Brownian motion and a compound Poisson process $X_t = B_t + \mu t + \sum_{i=1}^{N_t} Z_i$, this equation is considerably more complicated:

$$\begin{aligned} &\frac{f''(x)}{2} + \mu f'(x) - (\delta + \lambda) f(x) + \lambda \int_{-\infty}^{\infty} f(x+z)\nu(dz) = 0 \\ &f(x) = 1 \qquad \text{for} \qquad x \leq x_1 \\ &f(x) = 1 \qquad \text{for} \qquad x \geq x_2 \end{aligned}$$

where λ is the intensity of the Poisson process and $\nu(dz)$ is the distribution of the jumps.

1.4 Exercises

Exercise 1.2 If X_n is an additive or multipicative martingale, $F_n = X_1, \dots X_k$ is the information up to time n, and k is any number larger or equal to 1, show that

$$E[X_{k+n}/F_n] = X_n$$

Exercise 1.3 (Wald's martingale) Show that if Y_t is a Levy process with c.g.f. $c(u) = log(\mathbb{E}e^{uY_t})$, then $X_t = e^{uY_t - tc(u)}$ is a martingale.

Exercise 1.4 (Ross 6.13) Is the optional stopping theorem applicable to the martingale $X_n = \prod Z_i$, where Z_i take the values 2 and 0 with equal probability, stopped at the time T_0?

Exercise 1.5 The expected exit time

We may also find the expected exit time $\mathbb{E}_x T$ using martingales; this requires however using a rather nonobvious martingale: $M_n = X_n^2 - n$.

a) Show that M_n is indeed a martingale.

b) Which set of conditions should be used here to justify applying the optional stopping theorem? c) Applying the optional stopping theorem, show that:

$$t_x = E_x[min(T_L, T_K)] = (K - x)(x - L) \tag{45}$$

Notes: 1) The expected exit time equals the product of the distances from the initial capital x to the bounds of the interval.

2) Letting $L \to \infty$ and $K = x+1$ we find that the expected duration of a game for winning just one buck (with no lower bound on the losses) is infinite, which is quite surprising, given that the game has big probabilities of ending quite soon.

Exercise 1.6 Show that if B_t is standard Brownian motion then $M_t = e^{\sigma B_t - \frac{\sigma^2}{2}t}$ is a martingale (Ross, Exercise 19, pg 555).

Exercise 1.7 Solve Exercise 21 from Stochastic Processes of Ross, pg 556.

Exercise 1.8 (Ross 6.2) If $X_n = \sum^n Z_i$ is a martingale, then $Var(X_n) = \sum Var(Z_i)$

Exercise ** 1.9 Solve Exercise 22 from Stochastic Processes of Ross, pg 556.

Exercise ** 1.10 Prove the law of conditional expectations:

$$E[X/Z_1, \ldots, Z_k] = EE[X/Y_1, \ldots, Y_j, Z_1, \ldots Z_k]$$

1.5 Solutions

Solution 1.1 The statement holds by definition if $n = 1$. For $n \geq 2$, consider first the case of additive martingales, when $X_{n+k} = X_k + \sum_{i=1}^n Z_i$, with Z_i being independent with mean 0. Then,

$$E[X_{k+n}/F_k] = E[X_k + \sum_{i=1}^{n} Z_{k+i}/F_k]$$
$$= X_k + E[\sum_{i=1}^{n} Z_{k+i}] = X_k$$

The case of multiplicative martingales is similar.

Solution 1.2 (Wald's martingale)

$\mathbb{E}[X_{t+s}/X_s] = \mathbb{E}[e^{u(Y_{t+s}-Y_s)+uY_s-(t+s)c(u)}/Y_s = e^{uY_s-sc(u)}\mathbb{E}e^{u(Y_{t+s}-Y_s)-tc(u)} = X_s\mathbb{E}e^{uY_t-tc(u)} = X_s$ and so X_t is a martingale.

Solution 1.3 We check first that X_n is a multiplicative martingale (since $\mathbb{E}Z_1 = 1$). The optional stopping theorem $\mathbb{E}_x X_{T_0} = x$ applied to the stopping time T_0 (without checking the conditions) would yield here a wrong conclusion, that $x = 0$, whereas the starting point x is arbitrary.

Of course, none of the alternative conditions provided for the theorem holds here. For example, condition (2) (which is the most widely applicable) does not hold since a martingale which may double its value an arbitrary number of time does not have bounded increments.

Note: This exercise is similar to the martingale doubling strategy.

Solution 1.4 a) To show that M_n is indeed a martingale we obtain first a formula for its increments:

$$M_{n+1} - M_n = (X_n + Z_{n+1})^2 - n - 1 - (X_n^2 - n) = 2Z_{n+1}X_n + Z_{n+1}^2 - 1 = 2Z_{n+1}X_n.$$

We check now the conditional expectation of the increments.

$$E[M_{n+1} - M_n|F_n] = E[2Z_{n+1}X_n|F_n] = 2X_nE[Z_{n+1}|F_n] = 0.$$

b) We apply now the Optional Stopping Theorem to the martingale $M_n = X_n^2 - n$. This martingale is not bounded below (since T can take arbitrarily large values), so we can't apply the first set of conditions; however, the second set of conditions is satisfied since the increments of M_n are bounded:

$|2Z_{n+1}X_n + Z_{n+1}^2 - 1| = |2Z_{n+1}X_n| \leq 2\max(|L|, K)$

and $\mathbb{E}T < \infty$ will be seen later by the result of this exercise (and a very circular reasoning!)

The Optional Stopping Theorem yields:

$$\mathbb{E}_x M_T = \mathbb{E}_x(X_T^2 - T) = X_0^2 = x^2 \tag{46}$$

Conditioning on the last state we get

$$\mathbb{E}_x(X_T^2 - T) = K^2\mathbb{P}\{X_T = K\} + L^2\mathbb{P}\{X_T = L\} - E_x T. \tag{47}$$

The probabilities of winning/losing for the martingale X_T were found before to be

$$P[X_T = K] = \frac{x - L}{K - L}$$
$$P[X_T = L] = \frac{K - x}{K - L}$$

Plugging these in (47) gives

$$K^2\frac{x - L}{K - L} + L^2\frac{K - x}{K - L} - E_x T = x^2$$

which after simplifying yields

$$t_x = E_x[min(T_L, T_K)] = (K - x)(x - L) \tag{48}$$

Solution 1.5 Ross, 19: We check the martingale property that

$$E[M_{t+s}|F_t] = M_t$$

Indeed,

$$E[M_{t+s}|F_s] = E[\exp\{\theta(B_t - B_s)|F_s] - (g(t) - g(s))\} = 1$$

Since $B_t - B_s$ is independent of F_s and has $N(0, t-s)$ distribution, and its moment generating function is $E[\exp\{\theta(B_t - B_s)\}] = e^{\theta^2(t-s)/2}$, we get $g(t) = \frac{\theta^2}{2}t.)$

Solution: 1.6 The case of additive martingales (when the increments are independent) is well known. For the general case we use

$$Var(X_n) = Var(X_{n-1}) + Var(Z_n) + EZ_n\ X_{n-1}$$

(Note that we know all variables have 0 mean). The last term eqals 0 since $EZ_n\ X_{n-1}) = E[Z_n\ X_{n-1}/F_{n-1}] = X_{n-1}E[Z_n/F_{n-1}] = 0$.

Solution 1.7 Since the collections of random variables $Z_1, \ldots, Z_k$ and $Y_1, \ldots, Y_j$ can be both viewed as single vector variables $Y = Y_1, \ldots, Y_j$, $Z = Z_1, \ldots, Z_k$, this is equivalent to showing that

$$E[X/Z] = EE[X/Y, Z] \tag{49}$$

We will only establish this for discrete random variables and in its simplest form stating that

$$E[X] = E\ E[X/Y] \tag{50}$$

(The apparently more general form (49) reduces to applying the (50) for each fixed value $Z = z$.)

Let us denote by p_x the probability that X takes a certain value x, so that $E[X] = \sum_x xp_x$. Let us denote by $p_{x,y}$ the joint probabilities and by $p_{x/y}$ the conditional probability that $X = x$ given that $Y = y$, so that $E[X/Y = y] = \sum_x xp_{x/y}$. Then,

$$\begin{aligned} E\ E[X/Y] = \sum_y p_y E[X/Y = y] &= \sum_y p_y (\sum_x xp_{x/y}) \\ = \sum_{y,x} xp_y p_{x/y} = \sum_{y,x} xp_{x,y} &= \sum_y xp_x \\ &= E[X] \end{aligned}$$

2 Ito's formula and Stochastic Differential equations

In this section we start with a new derivation of Ito's formula, which is simpler, though somewhat subtler. We proceed then to discuss a class of models called **diffusions** which greatly enhances our repertoire of models.

Ito's formula will be rederived by using the basic approximation (51) below for the squared increments $dB_t = B_{t+h} - B_t$ of Brownian motion.

2.1 The unusual magnitude of Brownian increments

The following approximation, which expresses the unusual magnitude of Brownian increments, is the cornerstone of stochastic differential equations:

$$(dB_t)^2 \approx h \tag{51}$$

To understand this relation, recall that $dB_t = B(t+h) - B(t)$ is a zero mean Gaussian random variable with variance h and thus has the same distribution as

$$dB_t = \sqrt{h} N_{0,1}. \tag{52}$$

Note: For small h, $\sqrt{h}$ is much larger than h and thus the increments of the Brownian motion after intervals of size h are huge! The same idea was used in approximating Brownian motion by a random walk which added after time steps of size h increments of $\pm D = \pm\sqrt{h}$. If for example $h = 10^{-2}$ then $\sqrt{h} = 10^{-1}$ and this will look very "zigzagy".

To understand (51) we note that by (52) it follows that the random variable $(dB_t)^2 = hN^2$ has expectation h

$$\mathbb{E}\left(dB(t)^2\right) = h.$$

Furthemore, it has variance $\text{Var}\,(dB_t)^2 = \mathbb{E}\left(hN^2 - h\right)^2 = h^2 \text{Var}\,(N^2 - 1)$ which is of order of magnitude h^2, much smaller than h. Thus, the variability of $(dB_t)^2$ around its expectation is negligible, which explains (51)

To see why this is an indicator of unusual size, note that for any smooth deterministic process we have $(df(t))^2 \approx h^2 f'(t)$ and thus the squared increments are in that case of order of magnitude h^2, much smaller than h.

In conclusion, the "zigzagginess" of Brownian motion is mirrored in the unusual magnitude of the square of the increments (51), which is of the same order than the first. A similar relation holds for Brownian motion with drift $X(t) = \mu t + \sigma B(t)$:

$$(dX_t)^2 \approx \sigma^2 h \tag{53}$$

This is obtained using the relation $dX(t) = \mu h + \sqrt{h} N_{0,1}$ where $N_{0,1}$ is standard normal, and truncating powers of h of order greater than 1.

We record next for future use some moments of the increments of Brownian motion $dB(t)$ and of the Brownian motion with drift $dX(t)$. To check them, recall that the standard normal variable has all odd moments 0 and the fourth moment equals 3.

$$\begin{array}{lclll} EdB(t) & = & 0 \quad E(dB(t))^2 = h & E(dB(t))^3 = 0 & E(dB(t))^4 = 3h^2 \\ EdX(t) & = & \mu h \quad E(dX(t))^2 = \sigma^2 h + \mu^2 h^2 & E(dX(t))^3 = \mu^3 h^3 + 3\mu\sigma^2 h^2 & \end{array}$$

Finally, we show the impact of the exceptional size of the squared increments $dB(t)$ on approximating differentials of arbitrary functions applied to Brownian motion:

$$f(B_{t+h}) = f(B_t + dB_t) \approx f(B_t) + f'(B_t)d\,B_t + \frac{f''(B_t}{2}(d\,B_t)^2$$

We kept above **both the first and the second** order terms in the Taylor expansion, unlike in standard calculus in which only the first term is used. A similar approximation is used for functions of Brownian motion with drift X_t.

$$f(X_{t+h}) \approx f(X_t) + f'(X_t)d\,X_t + \frac{f''(X_t)}{2}\,(d\,X_t)^2$$

In terms of the parameters μ, σ, and after taking expectations conditional on $X_t = x$ this becomes:

$$\begin{aligned} \mathbb{E}[f(X_{t+h})/X_t = x] &\approx f(x) + f'(x)\mathbb{E}dX_t + \frac{f''(x)}{2}\mathbb{E}(dX_t)^2 \\ &= f(x) + h(f'(x) + \frac{f''(x)}{2}) = f(x) + h(Gf)(x) \end{aligned}$$

We recognize our "mantra" from the previous two sections:

Starting at $X_0 = x$, the expected value of a function $f(X_h)$ after a small time interval h is given by the function at x + the size of the time interval h multiplied by a rate $(Gf)(x)$ where the differential operator G involves both a first order term $\mu f'$ corresponding to the deterministic part μt and a second order term $\frac{\sigma^2}{2} f''$ corresponding to the random part σB_t.

In the next section we extend this to a larger class of processes called diffusions.

2.2 Diffusions

Brownian motion, Brownian motion with drift and geometric Brownian motion are particular examples of **diffusions**.

These two simple additive processes are rarely appropriate for modeling real life, complex phenomena, whose increments are seldom independent. Much more common is the case when we require the increments of processes to satisfy certain relations, for example recursive relations connecting future values to past values. By using these relations, called difference equations for discrete processes and differential equations for continuous ones (which cannot usually be solved explicitly) we enlarge significantly the scope of phenomena we may model.

Definition: Diffusions are solutions of **Stochastic Differential Equation** of the form

$$dX_t = \mu_t dt + \sigma_t dB_t \qquad (SDE)$$

where B_t is a standard Brownian motion.

Thus, the evolution of a diffusion X_t is driven by two terms: a Brownian motion B_t and a drift term μ_t.

These two terms are separated because they represent different things. μ_t represents a fixed (usually smooth) function, i.e. the "classical" part of an ODE; thus, SDE's reduce to ODE's when $\sigma_t = 0$. The second term, $\sigma_t dB_t$ in an SDE models an unknown forcing term which gives rise to wild, nonsmooth (i.e. non differentiable) local fluctuations.

Thus, a diffusion is a process which behaves locally like Brownian motion with drift, but the **local drift** μ_t and **local standard deviation** σ_t are allowed to vary with time. The Brownian motion and Brownian motion with drift correspond to the cases $\mu_t = 0, \sigma_t = 1$ and $\mu_t = \mu, \sigma_t = \sigma$ (constant) respectively.

It may be shown that diffusions are Markovian processes with continuous (though nonsmooth) sample paths and in fact the converse is also true.

The nonsmoothness of diffusions implies first that they don't have derivatives to be manipulated. One can still work with their "differentials" $df(X_t) = f(X_{t+h}) - f(X_t)$ over small intervals h. However, great care will be needed when manipulating these differentials; while usually for smooth functions only the first derivative f' is needed to approximate a differential, in the case of diffusions one needs to use both the first two terms in Taylor's formula (unlike in the usual differential calculus where the first term is enough). This leads to the appearance of second order derivatives in our problems.

The need for this new approximation, called **Ito's formula**, can be traced down to the unusual magnitude of the diffusion increments $X_{t+h} - X_t$ (the "zigzaginess" of diffusions). A good place to start our study is to quantify this "zigzaginess" for the simplest diffusion, standard Brownian motion.

The above implies that the moments of a general diffusion satisfy:

$$\begin{aligned} E_x[dX_t] &= \mu_t h \\ E_x[dX_t]^2 &\approx \sigma_t^2 h \\ E_x[dX_t]^n &= O(h^2) \quad \text{for any } n \geq 3 \end{aligned}$$

Note again the "unusual" large size of the second moment, first met in standard Brownian motion, which expresses the "zigzagginess" of diffusions. This leads to the necessity to modify the classical rules of calculus when manipulating diffusions. We discuss next:

1. A modified rule for differentials of products.
2. Ito's formula for general diffusions.
3. ** The quadratic variation of diffusions.

2.3 The Differential of a Product of diffusions

While in usual calculus all products of two or more infinitesimals $(dt, dX_t, dY_t..)$ are neglected, in stochastic calculus the products of two difusion increments are not negligible.

Suppose X_t and Y_t are difusions with parameters μ_X, σ_X and μ_y, σ_y respectively, driven by two Brownian motions with correlation ρdt (i.e. $\mathbb{E}(dB_t^1)(dB_t^2) = \rho dt$).

$$dX_t = \mu_X dt + \sigma_X dB_t^1$$
$$dY_t = \mu_Y dt + \sigma_Y dB_t^2$$

Let $Z_t = X_t Y_t$ denote the product of the two diffusions. In standard calculus the product rule for differentials is $dZ_t = X_t(dY_t) + (dX_t)Y_t$, which follows from the identity:

$$dZ_t = X_t(dY_t) + (dX_t)Y_t + (dX_t)(dY_t)$$

Note that indeed, $dZ_t = X_{t+h}Y_{t+h} - X_tY_t = (X_t + (X_{t+h} - X_t))(Y_t + (Y_{t+h} - Y_t)) - X_tY_t = dX_tY_t + X_tdY_t + dX_t\ dY_t$.

In stochastic calculus however the term $(dX_t)\ (dY_t)$ may not be neglected, as it would be in usual calculus; it may however be approximated by its expectation, resulting in the **corrected product rule** for differentials:

$$d(X_tY_t) = (dX_t)\ Y_t + X_t\ (dY_t) + \mathbb{E}(dX_t)\ (dY_t)$$

In the case when $B^1 = B^2$ for example, we find by substituting the expressions for dX_t, dY_t that:

$$dZ_t = (Y_t\mu_X + X_t\mu_Y + \sigma_X\sigma_Y)dt + (Y_t\sigma_X + X_t\sigma_Y)dB_t$$

Exercise Compute $d(B_t^2)$ using the corrected product rule.

Solution:

$$d(B_t^2) = 2B_tdB_t + dB_t\ dB_t = 2B_tdB_t + dt$$

We will see next that we need to keep both the first two order terms in the Taylor expansion for differentials $df(X_t)$ of functions of a diffusion, which results in the so called

2.4 Ito's formula for general diffusions

We show now that due to the exceptional size of the increments of diffusions $X(t)$ we always need to keep the first two order terms in Taylor expansions of differentials $df(X_t)$:

$d\,f(X_t) \approx f'(X_t)d\,X_t + \frac{f''(X_t}{2}(d\,X_t)^2$

In terms of the parameters μ_t, σ_t this becomes:

$$\begin{aligned} d\,f(X_t) &\approx f'(X_t)d\,X_t + \frac{f''(X_t)}{2}\sigma_t^2\,d\,t \\ &\approx f'(X_t)\sigma_t d\,B_t + (f'(X_t)\mu_t + \frac{f''(X_t)}{2}\sigma_t^2)\,d\,t \end{aligned}$$

In integral form, this becomes:

$$f(X(T)) - f(X(0)) = \int_0^T f'(X_t)\sigma_t d\,B_t + \int_0^T (f'(X_t)\mu_t + \frac{f''(X_t)}{2}\sigma_t^2)\,d\,t$$

Ito's formula decomposes a function of a diffusion as a sum of a **stochastic integral** with respect to Brownian motion and a usual integral.

The first integral has to be carefully interpreted as a limit of discrete sums in which the function is **always evaluated at the leftpoint** of the discretization integrals. Doing that, it is easy to check that:

Lemma: Any stochastic integral $\int_0^T f(X_t)d\,B_t$ is a martingale with 0 expectation.

An important consequence of Ito's formula is that a function of a diffusion is again a diffusion, with readily identifiable drift and dispersion:

Lemma: If $X(t)$ is a diffusion with drift μ_t and dispersion σ_t, than $f(X(t))$ is also a diffusion, with drift $\bar{\mu}_t = f'(X_t)\mu_t + \frac{f''(X_t)}{2}\sigma_t^2$ and dispersion $\bar{\sigma}_t = f'(X_t)\sigma_t$.

The identification of the drift and dispersion parameters is of crucial importance in any computation with diffusions, and this explains the frequent use of Ito's lemma.

The quadratic variation of Brownian motion: $dB(t)^2$ approximately equals its expectation, in the sense that its variance converges to 0 when $h \to 0$. The relation

$$dB(t)^2 \approx h \tag{54}$$

has the unusual consequence that the quadratic variation of a Brownian motion is not zero. More precisely, the limit

$\lim_{h\to 0} \sum_{t=h}^{h\lfloor \frac{T}{h} \rfloor} (B(t) - B(t-h))^2 \approx \sum_{t=h}^{h\lfloor \frac{T}{h} \rfloor} h = h\lfloor \frac{T}{h} \rfloor \approx T.$

Since the quadratic variation of any smooth process (with bounded first derivative) is easily shown to be 0, this is yet another indicator of "global zigzaggines".

Technical note: Exponential Brownian motion is itself a diffusion, whose generator is $Gf(s) = \frac{\sigma^2}{2}\, s^2\, f''(s) + r\, s\, f'(s)$ (called Euler operator). However, applying directly the differential equations approach is not so useful here, since problems involving this generator are most efficiently solved by the substitution $s = \exp x$, which brings us to constant coefficients equations. In the context of our original stochastic model using the substitution $S = \exp X$ just means that we should try to work with the Brownian motion exponent rather than with the exponnetial Brownian motion stock process. Thus, it is preferable to attempt to reduce questions about exponential Brownian motionstock process to questions about the associated Brownian motion exponents, and solve those using the constant coefficients differential operator.

Quanto options are an unusual type of options on foreign assets, which yield at expiration some function (forward, call, put, binary) of the foreign asset, **expressed in the native currency** (bypassing the exchange process!). We have thus some foreign currency, say yens, which evolve as $Y_t = Y_0 e^{yt}$, a foreign asset evolving as $S_t = S_0 e^{g\ t + \sigma B_t}$, an exchange value given by exponential Brownian motion$E_t = E_0 e^{\mu t + \sigma_2 B_2(t)}$ of home currency units for each yen, and the home currency brings interest r, i.e. evolves as $B_t = B_0 e^{rt}$. We assume the two Brownian motions have correlation ρ. Under these conditions, it turns out that the value of a forward quanto is given by $S_0 e^{-\rho \sigma_1 \sigma_2 T}$ and the exchange option formula still holds, with this value of the forward. In this case however the forward may not be hedged statically anymore.

2.5 Exercises

Exercise 2.1 Identify the drift μ and dispersion σ of the following diffusions and indicate which are martingales:

1. $B(t) + 4t$
2. $B(t)^2 - t$
3. $t^2\ B(t) - 2 \int_0^t sB(s)ds$
4. $B(t)^3 - 3\ t\ B(t)$
5. * $B_1(t) + B_2(t)$
   where B_1, B_2 are independent Brownian motions.

Exercise * 2.2 Find the differential of $B_1(t)B_2(t)$ where B_1, B_2 are independent Brownian motions.

Exercise 2.3 Compute, using the integral form of Ito' formula, (i) $\mathbb{E}_0 B_t^2$, (ii) $\mathbb{E}_0 B_t^4$ and (iii) ** $\mathbb{E}_0 e^{uB_t}$.

Stationary distributions of one dimensional difusions

The stationary density of one dimensional difusions $V(t)$ with drift $\mu(v)$ and standard deviation $\nu(v)$ is easily computable as: $p(v) = k\frac{e^{s(v)}}{\nu^2(v)}$ where $s(v) = 2\int \frac{\mu}{\nu^2}du$ and k is the proportionality constant (this can be shown by finding an associated O.D.E. $\frac{\partial^2}{\partial v^2}[\frac{\sigma^2}{2}p(v)] - \frac{\partial}{\partial v}[\mu p(v)] = 0$ and solving it).

Exercise ** 2.4 The volatility of an asset process is believed to be modeled by: either an Orenstein Uhlenbeck defined by the SDE (a)

$$dV_t = a(c - V_t) + \sigma dB_t$$

(b) or by the SDE

$$dV_t = a(c - V_t) + \sigma V_t dB_t$$

(i) Find the formulas of the stationary densities in both cases up to the proportionality constant (i.e., do not determine the constant).

(ii) What is the main difference between the two models; which is preferable?

Exercise 2.5

Let B_t be a standard Brownian motion with $B_0 = 0$.

(a) Determine a function of one variable $g(t)$ so that the random process

$$M_t^\theta = \exp\{\theta B_t - g(t)\}$$

is a martingale with initial value 1 (θ may be any fixed number). [4]

Exercise 2.6 Approximating the value of Asian options

2.6 Solutions

Solution 2.1

1. $$d(B_t + 4t) = dB_t + 4dt.$$

   So $\mu = 4, \sigma = 1$.

2. Recall first that the first term $d(B_t^2)$ has been found (using Ito's correction) to be $d(B_t^2) = 2B_t dB_t + dB_t\, dB_t = 2B_t dB_t + dt$. Thus,

   $$d(B_t^2 - t) = 2B_t dB_t + dt - dt = 2B_t dB_t.$$

   So $\mu = 0, \sigma = 2B_t$. This is a martingale.

3.

$$d(t^2\ B_t - 2\int_0^t sB_s ds) = 2tB_t dt + t^2 dB_t - 2tB_t dt = t^2 dB_t.$$

(No Ito correction is necessary). So $\mu = 0, \sigma = t_2$. This is a martingale.

4. The diferential of the first term is by Ito' correction: $dB_t^3 = 3B_t^2 dB_t + 3B_t dt$. Thus,

$$d(B_t^3 - 3\ t\ B_t) = 3B_t^2 dB_t + 3B_t dt - (3B_t dt + 3t dB_t) = 3(B_t^2 - t)dB_t.$$

So $\mu = 0, \sigma = 3(B_t^2 - t)$. This is a martingale.

5.

$$d(B_1 + B_2) = (dB_1) + (dB_2) = \sqrt{2} d\tilde{B}.$$

So $\mu = 0, \sigma = \sqrt{2}$. This is a martingale.

Solution 2.2

By Ito's product correction, $d(B_1 B_2) = B_1(dB_2) + (dB_1)B_2 + (dB_1)(dB_2)$. This exercise raises the more general issue of how to approximate $(dB_1)(dB_2)$ when the two Brownian motions have correlation ρ. The answer is provided by the fact that Ito's correction replaces the product $(dB)(dB)$ by its expectation dt. It turns out that more generally, that Ito's correction replaces also products of different differentials by their expectation and thus

$$(dB_1)(dB_2) \approx \rho dt.$$

In the case of independent Brownian motions with $\rho = 0$ this term falls down (thus there is no correction and

$$d(B_1 B_2) = B_1(dB_2) + (dB_1)B_2$$

Solution 2.3

(i) Letting $f(x) = x^2$ and $g(t) = \mathbb{E}_0 B_t^2 = \mathbb{E}_0 f(B_t)$ we find by Ito's integral formula that

$$g(t) = g(0) + \mathbb{E}_0 \int_0^t df = \mathbb{E}_0 \int_0^t \frac{f''(B_s)}{2} ds = \mathbb{E}_0 \int_0^t \frac{2}{2} ds = t.$$

(ii) Letting $f(x) = x^4$ and $g(t) = \mathbb{E}_0 B_t^4 = \mathbb{E}_0 f(B_t)$ we find by Ito's integral formula that

$$g(t) = \mathbb{E}_0 \int_0^t df = \mathbb{E}_0 \int_0^t \frac{f''(B_s)}{2} ds = \mathbb{E}_0 \int_0^t 6\mathbb{E}_0(B_s^2) ds = \mathbb{E}_0 \int_0^t 6s ds = 3t^2.$$

For example, for $t = 1$ we find the fourth moment of the standard normal: $\mathbb{E}B_1^{=}\mathbb{E}N^4 = 3$.

(iii) ** Letting $f(x) = e^{ux}$ and $g(t) = \mathbb{E}_0 e^{uB_t} = \mathbb{E}_0 f(B_t)$ we find by Ito's integral formula that

$$g(t) = 1 + \mathbb{E}_0 \int_0^t df = \mathbb{E}_0 \int_0^t \frac{f''(B_s)}{2} ds = \mathbb{E}_0 \int_0^t \frac{u^2 \mathbb{E}_0 e^{uB_s}}{2} ds = \mathbb{E}_0 \frac{u^2}{2} f(s) ds,$$

which is an integral equation!

Differentiating yields $f'(t) = \frac{u^2}{2} f(2)$ which together with the initial condition $f(0) = 1$ yields $f(t) = e^{\frac{u^2 t}{2}}$, the well known moment generating of a Gaussian (more easily computed by completing the square).

Solution 2.4

(i) $s(v)$ for our two models is respectively:

a) $s(v) = 2 \int \frac{a(c-v)}{\sigma^2} dv = \frac{2a}{\sigma^2}(cv - \frac{v^2}{2})$ and

(b)$s(v) = k2 \int \frac{a(c-v)}{\sigma^2 v^2} dv = -\frac{2a}{\sigma^2}(\frac{c/v}{+} ln(v)$. In this case the density further simplifies to:

$$p(v) = kv^{-\frac{2a}{\sigma^2}} e^{-\frac{2ac}{\sigma^2 v}}$$

(ii) The first model predicts a Gaussian distribution, which can, even if with small probability, take negative values. The second model is preferable since its stationary distribution is concentrated on the positive numbers.

Solution 2.5

(a) By Ito's formula,
$dM_t = M_t(\theta dB_t - g'(t)dt) + M_t \frac{\theta^2}{2} = M_t(\frac{\theta^2}{2} - g'(t)dt) + M_t \theta dB_t$
To cancel the drift, g must satisfy $g'(t) = \frac{\theta^2}{2}$ and thus $g(t) = \frac{\theta^2}{2}\ t$ (the additive integration constant was taken 0 to yield the initial value 1). Thus, the martingale is

$$M_t^\theta = \exp\{\theta B_t - \theta^2 t/2\}.$$

(Note: This part may be also obtained by the definition of the martingale property: Let $F_t = \sigma\{B_s : \ s \le t\}$. For $s \le t$, we must have

$$E[(M_t/M_s)|F_s] = E[\exp\{\theta(B_t - B_s)|F_s] - (g(t) - g(s))\} = 1$$

Since $B_t - B_s$ is independent of F_s and has $N(0, t-s)$ distribution, and its moment generating function is $E[\exp\{\theta(B_t - B_s)\}] = e^{\theta^2 (t-s)/2}$, we get $g(t) = \frac{\theta^2}{2} t$.)

(b) Since any martingale must satisfy $EM_t = M_0 = 1$ we find that

$$E \exp\{\theta B_t\} = e^{\theta^2 t/2}.$$

The geometric Brownian motion may be represented as $e^{-\alpha t + \theta B_t}$. Thus, its expectation is $e^{(\theta^2/2 - \alpha)t}$. The expectation is quite large for large t, if $\theta^2 > 2\alpha$, so this investment has "potential". However, $\alpha > 0$ ensures that the stock would eventually go to 0, and thus the "potential" may only be realized by "diversifying" (using several stocks).

(c) The one dimensional Brownian motion reaches a.s. any point (is recurrent) and thus $T_a < \infty$ a.s. Also, $|M_{\min(t,T_a)}| \leq e^{\theta\, a}$ if $\theta > 0$; thus, we may apply the optional stopping theorem:
$$1 = M_0 = E[M_{T_a}] = E[e^{\theta\, a - \theta^2 T_a/2}].$$
Putting $\lambda = \theta^2/2$ gives the desired result.

(d) Differentiating the moment generating function of T_a with respect to λ and plugging $\lambda = 0$ we find $ET_a = \infty$.

(e) We condition on $T = t$; letting $f_T(t)dt$ denote the stopping time's density, and noting that the conditional distribution of B_T is $N(0,t)$ (since T is independent of B), we find
$$E[g(T)e^{\theta B_T}] = \int_t f_T(t)dt(g(t)Ee^{\theta B_t}) = \int_t f_T(t)dt(g(t)e^{\theta^2 t/2}) = E[g(T)e^{\theta^2 T/2}].$$

(f) Let $T_{\log h} = \{\inf\{t : W_t = \log h\}$ (note that $\log h > 0$). The expected value of the exchange-adjusted profit is
$$E[he^{-\alpha T + \theta B_T}] - 1 = E[he^{-\alpha T + \theta^2 T/2}] - 1 \quad \text{by (c)}$$
$$= he^{-(\log h)\sqrt{2\alpha - \theta^2}} - 1 \quad \text{by (b) with } a = \log h \text{ and } \lambda = \alpha - \theta^2/2 > 0$$
$$= h^{1-\sqrt{2\alpha-\theta^2}} - 1.$$

The expected profit is negative if $2\alpha > \theta^2 + 1$ and positive if $\theta^2 < 2\alpha < \theta^2 + 1$.

Note however that the positive "expectations" are marred by our knowledge that they will not be fulfilled ($\alpha 0$ implies that the process $e^{-\alpha t + \theta B_t}$ converges to 0 and since T has infinite expectation and so is likely to be large, we will end up probably holding almost nothing, despite the positive "expectations".)

(g) If $2\alpha - < \theta^2$ then we expect infinite profit, even though the exchange has downward drift! This is due to the fact that the expected time until selling is infinite.

More precisely, using $e^x > 1 + x$ we see that for any a and $\lambda < 0$ we have $E[e^{-\lambda T_a}] = \infty$, since $E[e^{-\lambda T_a}] > 1 + (-\lambda ET_a) = \infty$. Over a very long time, the expected value of the fluctuations of the Brownian motion can beat "shoulders to the mat" the negative drift! (However, this is not likely to actually happen).

3 Optimization of portfolios of Exponential Brownian motions

3.1 The evolution of the combined portfolio's value

The simplest model of a **market** is that of a finite collection of I assets (stocks, bonds,options, cash) whose prices are denoted as $S_i, i = 1, ...I$. The task of a portfolio manager over one period is to divide his current wealth W in proportions π_i to be invested in each asset so that the **composite portfolio** $\pi_i W, i = 1, ...I$ has a return with "favorable" distribution. We will explain in the next section what may be meant by "favorable". For example, we would like to have a large mean, but small variance and "tails"; since these goals however turn out to be contradictory, it is impossible to select a universally acceptable optimization goal.

Definitions:

- The **total returns** of the assets over one period will be denoted by $d\,S_i, i = 1, ...I$.
- The corresponding **rates of return** will be denoted by

$$R_i = \frac{d\,S_i}{S_i}.$$

Clearly, the total return of a composite portfolio containing $\pi_i W$ of the i'th asset is $d\,W = W(\sum_i \pi_i \frac{d\,S_i}{S_i})$. We will refer to this fundamental equation relating the return rate of a composite portfolio to the return rates of its components as

The Combined return (wealth) equation:

$$\frac{dW}{W} = \sum \pi_i \frac{dS_i}{S_i} \tag{55}$$

Note that $\frac{dW}{W}$ is precisely the return rate R of the combined portfolio (and $\frac{dS_i}{S_i}$ are the return rates R_i of the individual assets) and so we will also write this equation as:

$$\tilde{R} = \sum \pi_i R_i \tag{56}$$

where we denoted by $\tilde{R}$ the combined return $\frac{dW}{W}$.

3.2 Possible portfolio optimization objectives

In this section we discuss several possible portfolio optimization objectives. The winner for GBM portfolios will be presented in the section folowing this.

The first possible optimization objective to come to mind for selecting a portfolio is maximization of the expected mean return. This turns out to be inappropriate however without further qualification, since arbitrarily large expected mean returns can be achieved by using arbitrarily large loans.

This comes at the price of increasing the variance of the return, i.e. the "risk", as illustrated in the example below.

Example 1.1: Leveraged position The oldest and most famous portfolio optimization technique is to secure a good loan and invest the money in stocks. The method can achieve arbitrarily large expected returns, unfortunately at the price of increasing the risks.

Indeed, consider for example a market with two assets: a **riskless** (nonrandom) cash investment S_0 with 0 interest rate and a stock S_1 whose return rate maybe 0 or 1 with equal probability.

The returns equations are thus:

$$\begin{aligned} d\,S_0 &= 0 \\ \frac{dS_1}{S_1} &= R_1 \text{ where } R_1 = 0 \textit{ or } 1 \text{ with equal probability} \end{aligned}$$

Note that $\mathbb{E}R_1 = 1/2 = \sigma_{R_1}$. Let π denote the proportion invested in the stock (and $1-\pi$ the proportion in cash). The combined return rate is thus πR_1 and the expected value and standard deviation of the return rate are both $\frac{\pi}{2}$. If furthermore we can take an arbitrarily large loan, which is the same as there being no bounds on π, we see that we can achieve arbitrarily large expected returns, but the standard deviation becomes also arbitrarily large.

While in this example clearly we should take advantage of all the leverage we can get, in general the acompanying increases in risk may not be acceptable. For this reason, it is impossible to choose universally good optimization objectives. Instead, the preferences of the investor (his tolerance to risk) have to be taken into account also.

The second optimization objective to come to mind is minimizing the variance. Note however that this can simply be achieved by holding cash only. Since it so not possible simultaneously to maximize the expected returns and minimize their variance, Markowitz proposed a tradeoff to be be discussed in the appendix.

In the next section we will explain another objective which is particularly convenient for portfolios of Brownian motions: maximizing the expected logarithm of the wealth $\mathbb{E}ln(W_T)$, which as we will explain is also tied to the long run growth of the portfolio.

3.3 Maximization of long run growth

In this section we explain that maximizing almost surely the long run growth of an investment is tantamount to maximizing the expectation of the logarithm of the final wealth $\mathbb{E}ln(W_T)$.

Let us note first that **Wealth is a multiplicative process,** i.e. if we denote by W_1 the wealth at the end of a period, then

$$W_1 = W_0 + W_0\tilde{R} = W_0\tilde{Y}$$

with the **total yield** of each dolar being $\tilde{Y} = 1 + \tilde{R}$. After T periods, the wealth becomes:

$W_T = W_0 \prod_{i=1}^{T} Y_i$.

We will state now a formula for the long run growth of multiplicative processes (with i.i.d. factors).

Lemma 3.1 Long run growth Suppose the yields Y_i of a portfolio are i.i.d. random variables. Then, the total value $V(T) = \prod_{i=1}^{T} Y_i$ after time T of one unit of currency, will be approximatively $e^{T\mathbb{E}ln(Y)}$ **pathwise!**

Proof sketch: We take logarithms: $ln(V(T)) = \sum_i ln(Y_i)$, so that we can apply the law of **large numbers** for additive processes. By the latter, the sum of the logarithms will be closer to $T\mathbb{E}ln(Y)$ pathwise, yielding the result.

Note: The **pathwise growth** return rate $\mathbb{E}ln(Y)$is considerably smaller than the **expected** return rate $ln(\mathbb{E}Y)$. In finance, positive expectations often go together with a.s. bankruptcy!

Example: Dynamic rebalancing strategies A portfolio made of I assets with "stationary" yields $Y_k^{(i)}$, managed so that the proportions π_i of the various investments is constant over time is one example of a stationary multiplicative process. Indeed, since the yields equal the returns plus 1 a "combined yield" equation similar to the "combined return" equation holds for composite portfolios:

$$Y_k = \sum_i \pi_i Y_k^{(i)}$$

and this does not change with time for dynamic rebalancing strategies.

Of course, such investment policies are very simple minded, but they serve as a useful starting point for mathematical modelling.

The exercise below illustrates the superior growth rates as well as the decrease in risk (variance) which can be achieved by combining several investments and using the simple rebalancing technique described above.

Exercise 1: Consider two stocks evolving as standard geometric random walks, i.e. by multiplication with $Y_n = e^{Z_n}$ and $Y'_n = e^{Z'_n}$ respectively, where Z_n, Z'_n are independent r.v.'s which equal ± 1 with equal probability. Thus, the respective values at time n are $S_n = \prod^n Y_i, S'_n = \prod^n Y'_i$.

Find the mean and variance of

(a) The portfolio S_n

(b) The "diversified", but not "rebalanced" portfolio $T_n^{(0)} = S_n/2 + S'_n/2$.

c)The portfolio W_n which is rebalanced half half after each time unit, so that its value at time n is given by

$W_n = \frac{W_{n-1}}{2} Y_n + \frac{W_{n-1}}{2} Y_n'$.

c) Find the long run growth of S_n and of W_n.

Solution: a) $\mathbb{E}S_n = (\mathbb{E}Y_1)$ with $\mathbb{E}Y_1 = \frac{e+e^{-1}}{2}$.

$Var(S_n) = \mathbb{E}(S_n^2) - (\mathbb{E}S_n)^2 = (\mathbb{E}Y_1^2)^n - \mathbb{E}Y_1^{2n}$, with $\mathbb{E}Y_1^2 = \frac{e^2+e^{-2}}{2}$.

b) $\mathbb{E}T_n = \mathbb{E}S_n$. $Var(T_n) = \frac{2Var(S_n)+2cov(S_n,S_n')}{4} = \frac{Var(S_n)}{2}$.

c) Note that $W_n = \prod_1^n U_i$, where $U_i = \frac{Y_i+Y_i'}{2} = e, e^{-1}$, or$\frac{e+e^{-1}}{2}$ with probabilities $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$. Thus, $\mathbb{E}U_1 = \mathbb{E}Y_1$ and

$$
\begin{aligned}
\mathbb{E}U_1^2 &= \frac{e^2+e^{-2}+2(\frac{e+e^{-1}}{2})^2}{4} \\
&= \frac{1}{4} + \frac{3(e^2+e^{-2})}{8}.
\end{aligned}
$$

Like in a), we get $\mathbb{E}W_n = \mathbb{E}S_n$ and $Var(W_n) = \mathbb{E}(W_n^2) - (\mathbb{E}W_n)^2 = (\mathbb{E}U_1^2)^n - \mathbb{E}Y_1^{2n} = (\frac{1}{4} + \frac{3(e^2+e^{-2})}{8})^n - \mathbb{E}Y_1^{2n}$.

3.4 The relation between the long run growth rate and the expected rate of return for Geometric Brownian motion

In this key section we describe the relation between our winner for optimization objective, the long run growth rate, and the expected rate of return of a GBM, which is easier to manipulate.

Consider an asset whose total yield is geometric Brownian motion $S(t) = e^{gt+\sigma B_t}$ (g being the growth rate). By Ito's formula we may establish a relation between the growth rate of return and the expected rate of return.

Lemma 3.2 Equivalence of GBM formula and linear SDE a)A geometric Brownian motion $S_t = e^{gt+\sigma B_t}$ satisfies the linear stochastic differential equation (SDE)

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$$

where $\mu = g + \frac{\sigma^2}{2}$. (Note that $\mu dt = \mathbb{E}\frac{dS_t}{S_t}$, and so μ is the **expected** rate of return).

b) Viveversa, the solution of the linear SDE $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ has to be a geometric Brownian motion $e^{gt+\sigma B_t}$ where $g = \mu - \frac{\sigma^2}{2}$.

Proof: (a) Letting $f(t,x) = e^{gt+\sigma x}$ so that $S_t = f(t, B_t)$ we find by Ito's formula that

$$dS_t = \frac{\partial}{\partial t} f dt + \frac{\partial}{\partial x} f dB_t + \frac{\frac{\partial^2}{\partial x^2} f}{2}(dB_t)^2 = gS_t dt + \sigma S_t dB_t + \frac{\sigma^2}{2} S_t dt = S_t(\mu dt + \sigma dB_t)$$

Recall that all the quantities above have financial interpretations: $dS(t)$ is the return over a period of time dt of the asset $S(t)$, $\frac{S(t+dt)}{S(t)}$ is the rate of return (per currency unit) and μ is the expected rate of return (per unit of time). The parameter μ has a second interpretation:

Lemma 3.3 The expected total yield of one currency after time t is $e^{\mu\, t}$.

Indeed, $\mathbb{E}S_t = \mathbb{E}e^{gt+\sigma B_t} = e^{gt+\frac{\sigma^2 t}{2}} = e^{\mu\, t}$.

The next lemma shows that the pathwise rate of return of geometric assets, to be called growth rate is only g, less than the expected rate of return μ.

Lemma 3.4 Pathwise, $S_t \approx e^{gt}$

This is a consequence of the law of large numbers by which the additive process $gt+\sigma B_t \approx gt$ pathwise.

Paradoxically, the approximate pathwise rate of growth g of geometric Brownian motion is less than the expected rate of return μ. It is quite possible that such a process will go to 0 on most simulations (pathwise), but its expectation will go to ∞. This paradoxical behavior common to all multiplicative processes is explained by the fact that in the "one in a hundred chance" that the process will not go down but up the expected increase may more than counterbalance the losses in the other "ninetynine in a hundred chance" that the process goes down.

In conclusion, the main characters of portfolio optimization, the expected return rate per unit time μ and the growth rate per unit time g are related in the case of geometric Brownian motion by the formula:

$$\mu = g + \frac{\sigma^2}{2}. \tag{57}$$

3.5 The optimum growth portfolio with one GBM asset

We consider now markets with two assets only, one of which is a riskless asset $S_0(t) = e^{rt}$. The SDE's for the returns of the assets are thus:

$$\frac{dS_0}{S_0} = rdt$$
$$\frac{dS_1}{S_1} = \mu dt + \sigma dB_t$$

The combined return rate of a portfolio with π in the risky asset and $1-\pi$ in the riskless asset is:

$$\frac{dW}{W} = (1-\pi)\frac{dS_0}{S_0} + \pi\frac{dS_1}{S_1} = rdt + \pi(\mu - r)dt + \pi\sigma dB_t = \tilde{\mu}dt + \pi\sigma dB_t \tag{58}$$

where we denoted by $\tilde{\mu} = r + \pi(\mu - r)$ the expected return rate of the combined portfolio. The combined portfolio's volatility is $\tilde{\sigma} = \pi\sigma$; thus, the growth rate is:

$$\tilde{g} = \tilde{\mu} - \frac{\tilde{\sigma}^2}{2} = r + \pi(\mu - r) - \frac{1}{2}\pi^2\sigma^2.$$

Differentiating with respect to π we find that the optimum proportion of the risky asset π^* is given by:

$$\pi^* = \sigma^{-2}(\mu - r) \tag{59}$$

and the optimum growth rate is:

$$g^* = r + \frac{1}{2}(\frac{\mu - r}{\sigma})^2 = r + \frac{1}{2}\theta^2 \tag{60}$$

where we put $\theta = \frac{\mu - r}{\sigma}$. This quantity will turn out to appear in several formulas from now on. It may be interpreted as the **relative efficiency** of the stock S_1 with respect to the riskless asset or as the standard deviation $\pi^*\sigma$ of the optimal composite portfolio. It came to be known in the literature as the **market price for risk**.

Note: The maximum growth problem for a geometric Brownian motion asset above is precisely the same as the one period Markowitz problem with Lagrange multiplier $\lambda = \frac{1}{2.}$

Finally, the maximum achievable wealth obtainable by dynamic rebalancing may be characterized either as

$$W_t = e^{g^*t + \theta B_t} = e^{rt + \frac{1}{2}\theta^2 t + \theta B_t}$$

or as the solution of the SDE

$$\frac{dW}{W} = \mu^* dt + \theta dB_t \tag{61}$$

3.6 ** Optimization of portfolios of several Geometric Brownian motions

We will show now that a portfolio made up of several geometric brownian motion assets in constant proportions π_i is also a Geometric Brownian motion, with expected rate of return and standard deviation which depend linearly of π_i.

Note: the results below are formulated in a quite general form. They refer to a vector of I assets $\frac{dS(t)}{S(t)} = (\frac{dS_i(t)}{S_i(t)}, i = 1, ..., I)$ modeled by geometric brownian motions with SDE's given by

$$\frac{dS_t}{S_t} = \mu dt + \Sigma d\bar{B}_t, \tag{62}$$

which are driven by an arbitrary number of Brownian motions. The matrix Σ maybe in general rectangular. The matrix $A = \Sigma'\Sigma = \sigma_{i,j}$ can be shown to give the covariances of the assets (at time $t = 1$) and is thus to some extent observable, while the matrix Σ isn't. It turns out that in the case of rectangular Σ with full rank a certain very strong statement about replicating derivatives called **completeness** can be made, which is for sure not met in practice. Thus, for realism it is important to allow in modeling for the case of rectangular Σ with the number of sources of uncertainty (Brownian motions) exceeding the number of assets.

Note: some of the equations below are redundantly expressed both in concise matrix notation and in explicit summation notation, to accomodate all tastes.

Lemma 3.5 Combined GBM If a set of assets are modeled by a vector of I geometric brownian motions $\frac{dS(t)}{S(t)} = (\frac{dS_i(t)}{S_i(t)}, i = 1, ..., I)$ with SDE's given by

$$\frac{dS_t}{S_t} = \mu dt + \Sigma d\bar{B}_t, \tag{63}$$

where $\bar{B}_t$ is a vector of independent standard Brownian motions (and $\mu, \Sigma'\Sigma = \sigma_{i,j}$ represent the expected returns and the covariances of the returns R_i), then the combined dynamically rebalanced portfolio with constant weights π_i, i.e. the portfolio described by

$$\frac{dW}{W} = \sum \pi_i \frac{dS_i}{S_i} = \pi' \frac{dS_t}{S_t} \tag{64}$$

is also a geometric Brownian motion, with parameters

$\tilde{\mu} = \pi'\mu = \sum_i \pi_i \mu_i$, $\tilde{\sigma}^2 = \|\pi'\Sigma\|^2 = \pi'\Sigma\Sigma'\pi = \pi' A \pi = \sum_{i,j} \pi_i A_{i,j} \pi_j$.

Note: In the case of one asset discussed in the previous section this simply means that $\tilde{\sigma} = \pi\sigma$.

Proof Plugging the SDE formula for S_t (63) in the combined wealth equation (64) we find

$$\frac{dW}{W} = \pi' \frac{dS_t}{S_t} = \pi' \mu dt + \pi' \Sigma d\bar{B}_t = \tag{65}$$

Now $\pi'\mu = \tilde{\mu}$ and from the well known fact that a mixture of independent Gaussian random variables is also Gaussian with variance equal to the sum of the variances we find that the process $\tilde{\Sigma} d\bar{B}_t$ has the same distribution as $\tilde{\sigma} d\tilde{B}_t$ where $\tilde{B}_t$ is a standard one dimensional Brownian motion and $\tilde{\sigma}^2 = \|\pi'\Sigma\|^2$.

This simple Lemma leads immediately to a formula for the growth rate of mixtures of geometric Brownian motions. Indeed, we only have to plug in the relation $\tilde{g} = \tilde{\mu} - \frac{\tilde{\sigma}^2}{2}$ the formulas for $\tilde{\mu}, \tilde{\sigma}$ given above.

Note The results of the previous section extend immediately to the vector case of I risky assets. Denoting by π the vector of proportions of the risky assets, by $\mu - r$ the vector of excess expected returns of the risky assets over the riskfree rate, by Σ the matrix describing the linear dependence of the stock returns on the Brownian motions (sources of uncertainty) and by θ the vector of the market prices for risk $\bar{\theta} = \Sigma^{-1}(\mu - r)$ we have:

Lemma 3.6 (The Merton portfolio)

(a) The optimum vector of proportions of risky assets π^* satisfies:

$$\Sigma'\pi^* = \theta$$

(b) The optimum growth rate is:

$$g^* = r + \frac{1}{2}\|\theta\|^2$$

(c) The maximum achievable wealth is

$$W(t) = e^{rt+\frac{1}{2}\|\theta\|^2 t+\theta \bar{B(t)}}$$

where $\bar{B(t)}$ denotes a vector of independent standard Brownian motions.

4 Risk neutral valuation in Exponential Brownian motion markets

The fundamental question about derivatives is what is the current (present) value of a derivative which pays some function $f(S_t)$ at a later time, that is how much should people pay now for future prospects.

The Black-Scholes result lead to what is nowadays known as the **RN valuation** principle which put the focus on **risk neutral** processes. These are processes whose expectations increase as if they were riskless, or whose present value doesn't change, i.e.

$$\mathbb{E}\ S_t = S_0 e^{rt}$$

Briefly put, it says that the the fair price (present value) for any future random claim $H(S_T)$ contingent on an asset price S_T is given by its discounted expectation

$$\mathbb{E}_Q\ e^{-rT} H_T$$

where r is the risk free interest rate of the market and

- Q is a **modified measure** with respect to which asset values have expectations which increase as if they were riskless, i.e.

$$\mathbb{E}_Q\ S_t = S_0 e^{rt}$$

- Q is **close in some sense to the original measure**

There exist two (sometimes equivalent) methods to answer this question:

1. Valuation as the **initial value of a replicating portfolio.** The current value of a future claim should equal the initial amount necessary to set up a "replicating portfolio", i.e. a portfolio whose final value equals (or approaches as much as possible) the final claim.

2. Valuation as the **"risk neutral" expected value of the final claim.**

The first method is an outcome of the crucially important fact that fund managers have to "hedge" risk.

The second method is an outcome of the observation made over the last 30 years that often the answer to various hedging problems may be expressed as the expectation of the final claim with respect to certain types of measures called **risk neutral**, which are related but different from the measure governing the evolution of the asset process. Often, as in the cases of binomial and GBM markets, risk neutral valuation leads to simple "cookbook" recipes, like the **risk neutral drift modification rule** for GBM's.

Below, after briefly discussing replication, we state the **risk neutral drift modification rule** for GBM's, as well as its reinterpretation as a **"discounted" value of the final**

claim with respect to the optimal performance achievable via portfolio optimization. Unfortunately, the equivalence between between the simple cookbook recipe and the intuitively plausible discounted form can only be justified via rather sophisticated mathematics, the "Cameron-Martin-Girsanov theorem". Finally, we show that optimal hedging in binomial and multinomial markets leads to risk neutral valuation and state this general general principle for arbitrary markets.

There are two types of markets (called "complete" markets) which have been shown to allow perfect replication.

- "Binomial" discrete market models in which at each moment in time the stock can choose to change only among two possible values.
- Continuous time difusion models(the restrictive assumptions here being that of continuous trading and that of absence of jumps).

The initial value v_0 of a replicating portfolio in a complete market is a much sounder basis for pricing options than expectations based on statistical models. However since "complete" markets exist only on paper, the situation in reality is not that clear cut, as witnessed by the crash in 1998 of some major derivative firms.

4.1 Speculator and "risk neutral" valuation in GBM markets

In the exercises below we compute expectations of various claims $\mathbb{E}h(S_t)$ for a GBM asset S_t. To stress that unlike "replication", expectation under a statistical model is not a sound basis for the pricing of options, we refer to it as "speculator expectation".

The good name of the "expectation" will however be redeemed later, when it will turn out that the replicating price v_0 itself is an expectation, but not under the original statistical model ("measure") for S_t. For example, in the case of GBM markets $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ the expectation will be for a new GBM model with modified drift:

$$\frac{dS_t^*}{S_t^*} = rdt + \sigma dB_t$$

where r is the risk free interest of the market. Equivalently, the growth rate of S_t^* is $r - \frac{\sigma^2}{2}$. This "cookbook" recipe is referred to as:

The Risk neutral drift modification rule: a) Given that the value of an asset at time $t = 0$ is $S_t = s_0$, the initial value necessary to start a replicating portfolio of a future claim $C_T = f(S_T)$ in a GBM market is given by

$$v_0 = \mathbb{E}_{s_0} e^{-rT} f(S_T^*)$$

b) Given that the value of an asset at time t is $S_t = s$, the necessary value the replicating portfolio of a future claim $C_T = f(S_T)$ has to have at time t is given by

$$v_{t,s} = \mathbb{E}_{\{t,S_t=s\}}[e^{-r(T-t)} f(S_T^*)/F_t]$$

Corollary: The Black Scholes equation

The function $v_{t,s}$ satisfies the partial diferential equation

$$\frac{\sigma^2 s^2}{2}\,\frac{\partial^2}{\partial s^2}v + rs\,\frac{\partial}{\partial s}v - r\,v = \frac{\partial}{\partial T}v$$

with final condition $v(T,s) = f(s)$.

This follows from the general recipe providing the PDE for a discounted exponential (see Section 3.5). Note however that the Markov dynamics used is not that of the original process S_t, but that of the modified process S_t^*.

In the next section we show that risk neutral valuation ma be interpreted as taking the expectation of the final claim $H(S_T)$ discounted with respect to the manager's discount Z_T (called **risk neutral discount factor**) which is the reciprocal of the optimum yield obtainable from a currency unit via portfolio optimization.

4.2 Pricing through discounting by the portfolio manager's performance

Recall that the optimum wealth achievable by a manager who can shuffle back and forth his money between the stock S_t and a riskless asset with interest rate r is

$$W_t = e^{rt+\theta B_t+\frac{1}{2}\theta^2 t}$$

The reciprocal process $W_t^{-1} = e^{-rt}Z_t$ where $Z_t = e^{-\theta B_t-\frac{1}{2}\theta^2 t}$ will turn out to play a fundamental role in determining the current value of **derivative contracts**.

We will suggest now a heuristic method of pricing which turns out to get the right answer for a wrong reason!

Consider first the simpler case of some pension plan which will deliver a single lump payment K at time t in the future. The pension plan is very conservative and may invest our payment v_0 only in the riskless investment with fixed interest rate r. Let $W_c(t) = e^{rt}$ denote the value of one currency unit at time T, by this conservative investment method. The payment to be requested initially will obviously be $\frac{K}{W_c(T)} = Ke^{-rt}$. Suppose now the payment $K$is replaced by a random claim $H_T = f(S_T)$ which depends on the final value of some stock! The pension plan now turned into insurance company might come up with the price $\mathbb{E}e^{-rt}H_T$ "in despair of the unknown" but this would have disastruous consequences due to the risk of not being able to meet the final claim, and as is well known, repeated risk leads to disaster (the law of large numbers is not good enough for insurance companies).

Suppose now we allow the insurance company to try and hedge the future claim by investing optimally in a mixture of the stock S_t and the riskless investment and suppose for a moment that the company has also an astrologuer who knows with certainty the future moves of the Brownian motion (and thus also what H_T and W_T will be. Obviously, the fair price for the claim would then be $\frac{H_T}{W_T}$ (to be presently invested with the company's portfolio manager who will deliver the claim at time T.)

Now the astrologuer leaves. The company can only find recourse in the "despair of the unknown" price

$$v_0 = \mathbb{E}\frac{H_T}{W_T}.$$

which should probably lead to disaster. At the last moment however they hire a financial engineer who saves them by pointing out that the price they charged was actually right. With precisely that price, he can assure the delivery of the claim by "hedging" it (instead of using dynamic rebalancing proportions like the portfolio manager).

So, for some misterious reason, the "despair" price $v_0 = \mathbb{E}\frac{H_T}{W_T} = \mathbb{E}e^{-rt}H_t Z_t$ turns out to be the same as the replicating price. The process Z_t which may be viewed as an extra discount factor appearing due to our use of optimal investing instead of conservative investing, i.e. a **manager's extra discount**, has some quite interesting properties:

Exercises

Exercise 4.1 Show that Z_t is a martingale.

Exercise 4.2 Show that $S_t Z_t$ is a martingale.

Exercise 4.3 Assuming $\theta > 0$, find the probability $\mathbb{P}\{Z_t \leq 1\}$ that the extra discount factor is smaller than 1 (that is that the portfolio manager using the optimal constant proportions is doing beter than the market).

Solutions

4.1 Z_t is a geometric Brownian motion with $g = -\frac{\sigma^2}{2}$.

4.2 $S_t Z_t$ is also a geometric Brownian motion with $\tilde{g} = g - \frac{\theta^2}{2} = \mu - \frac{\sigma^2}{2} - \frac{\theta^2}{2}$ and it may be checked that $-2\tilde{g}$ equals the square of the combined volatility $\sigma - \theta$. Alternatively, it is enough to show that the SDE satisfied by $S_t Z_t$ contains no drift term. Indeed, plugging $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ and $\frac{dZ_t}{Z_t} = -thdB_t$ in Ito's product rule we find that

$$\frac{d(S_t Z_t)}{S_t Z_t} = (\mu - \sigma\theta)dt + (\sigma - \theta)dB_t = rdt + (\sigma - \theta)dB_t = (\sigma - \theta)dB_t$$

4.3 $\mathbb{P}\{Z_t \leq 1\} = \mathbb{P}\{e^{-\theta B_t - \frac{\theta^2}{2}t} \leq 1\} = \mathbb{P}\{e^{-\theta\sqrt{t}N} \leq e^{\frac{\theta^2}{2}t}\} = \mathbb{P}\{-\theta\sqrt{t}N \leq \frac{\theta^2}{2}t\} = \mathbb{P}\{N \geq -\frac{\theta\sqrt{t}}{2}\} = \Phi(\frac{\theta\sqrt{t}}{2})$.

Note 1) Since a martingale satisfies $\mathbb{E}Z_t = Z_0$ and $Z_0 = 1$, we see that the manager's extra discount has expectation 1, which is somewhat misleading. The manager achieves an extra performance $W_t e^{-r_0 t} = e^{\frac{1}{2}\nu^2 t + \nu B_t}$ with positive growth rate $\frac{1}{2}\nu^2$ over the market, and so he will end up usually above the market (see Exercise 3 above), in which case the extra discount will be smaller than 1 (usually close to 0). However in the rare cases when he fails, the "extra" wealth he obtains $W_t e^{-r_0 t}$ will be smaller than 1, in which case the extra discount Z_t may be quite larger than 1 and mask in the arithmetic average the usual good management performance.

2) The second exercise shows that a GBM stock price discounted by the manager's performance has a "balanced" martingale distribution (called thus since its increments have mean 0, with positive increases counterbalancing on the average the negative decreases). (The distribution for arbitrary functions $f(S_t)$ obtained by multiplying them by Z_t is also known as the **risk neutral distribution**).

In the next section we state the basic result of derivative pricing: in certain markets called **complete** in which the risk neutral measure is unique, the initial value v_0 as well as the later value v_t of a (liquid) claim may be computed by taking the expectation of the final claim $H(S_T)$ discounted with respect to the unique **risk neutral measure**).

4.3 The fundamental theorem of derivative pricing

Consider a GBM market satisfying the SDE

$$\frac{dS_t}{S_t} = \mu dt + \Sigma d\bar{B}_t$$

in which the number of traded stocks (the dimension of S_t) equals the number of sources of uncertainty (the dimension of $\bar{B}_t$) and the matrix Σ has maximal rank. Let $H_T = f(S_T)$ denote any claim at time T contingent on the state of the market S_T.

We will assume also the availability of a riskless cash investment with fixed interest rate r. The value after time t of one currency unit invested in the riskless investment is thus e^{rt}. It becomes convenient then to measure all the asset values with respect to the cash investment, i.e. to use the cash investment as an artificial currency, called "numeraire". The "numeraire" value (or discounted value) of the any asset whose value is X_t is given by $\frac{X_t}{e^{rt}}$. The effect of this transformation is that the "numeraire" price of the cash investment becomes constant, which is tantamount to assuming $r = 0$.

For this reason it is a good idea to ignore at the first reading of the folowing statements the effect of the nonzero interest rate r, since in the "numeraire" world $r = 0$; to convert back to the original currency is as easy as multiplying by e^{rt}.

Definition: A replicating portfolio is a portfolio managed as follows:

a) The initial value of the portfolio (to be charged to the buyer) is

$$v_0 = e^{-r\,T}\mathbb{E}H_T Z_T$$

where $Z_t = e^{-\theta\bar{B}_t - \frac{\theta^2}{2}t}$, $\theta = \Sigma^{-1}(\mu - \bar{r})$ and $\bar{r}$ denotes a constant vector with all components r.

b) The value of the portfolio at time $t < T$ should be

$$V_t = V(t, S_t) = e^{-r\,(T-t)}\mathbb{E}[H_T Z_T / F_t]$$

c) The number of stock units to be held at time t should be $\Delta_t = \frac{\partial V_t}{\partial S_t}$. (Thus, to make up for the difference between the value V_t needed and the part which has to be kept in stock $\Delta(t)S(t)$ an additional loan $L(t) = V_t - \Delta_t S_t$ has to be taken.)

Definition: A portfolio with value V_t is selffinancing if its discounted value satisfies an SDE of the form

$$dV_t e^{-rt} = \varphi dS_t$$

where φ is the number of stock units held (i.e., the only change in the discounted value comes from "capital gains"). comes

Theorem 4.1 (**Exact replicating of derivatives in GBM markets**)

The replicating portfolio described above is selffinancing and ends up equalling at time T the value of the claim exactly, with no risk (with probability one).

Notes: 1) This result shows the surprising fact that the mere knowledge of the optimal growth $W_t = e^{rt} Z_t^{-1}$ achievable by portfolio optimization allows one to **replicate exactly any posssible claim**.

2) In the case of call options the expectation $V_t = e^{-r\,(T-t)}\mathbb{E}[(S_t - K)_+ Z_T / F_t]$ may be computed exactly, ending in the Black Scholes formula.

$$V(t) = S(t)\Phi(s(t)) - K\Phi(l(t))$$

This formula reveals immediately the hedging strategy which is to keep $\Phi(s(t))$ units of stock and a loan equal to a proportion of $\Phi(l(t))$ out of the final payment of K, without the need to compute the Δ_t by partial derivatives (we may check that indeed $\frac{\partial V}{\partial S} = \Phi(s)$, since $S\varphi(s) = K\varphi(l)$, where φ is the standard normal density).

3) An interesting fact which turns out to be quite significant is that the "extra" potfolio manager discount Z_t has the property that $S_t Z_t$ is a martingale (See Exercise 6.5). This fact will be generalized int he next section, in which we establish also the equivalence between the "drift modification" recipe and the "discounted value" formulations.

4.4 ** The Cameron-Martin-Girsanov theorem

Exercise 4.4 (**Cameron-Martin-Girsanov**) Suppose S_t satisfies $\frac{dS_t}{S_t} = \mu dt + \sigma dB_t$, let Z_t be an exponential martingale given by $\frac{dZ_t}{Z_t} = -\theta dB_t$ for some θ and let S_t^* satisfy the SDE with modified drift $\frac{dS_t^*}{S_t^*} = (\mu - \sigma\theta)dt + \sigma dB_t$. Show that for any exponential function $f(x) = e^{ux}$ the processes described by the formulas

1. $f(S_t)Z_t$ and
2. $f(S_t^*)$

satisfy the same SDE.

Note: This is the main step in establishing the Cameron-Martin-Girsanov theorem which states that any expectation of the modified process S_t^* under the original measure coincides with the "discounted" (weighted) expectation of the original process S_t. In other words, discounting the expectation by may be replaced by modifying the drift of the original process. Thus, in practical calculations it is not necessary to use the extra discount Z_t; it is more convenient instead to modify the drift of the process. Also, since we have converted the extra discounted expectations to usual expectations, we may conclude that the value of a claim at time t satisfies the PDE described at the end of section 3.5.

Theorem 4.2 (**The Cameron-Martin-Girsanov change of drift**) a) For any function $f(S_T)$, the "extra discounted" expectation $\mathbb{E}e^{-rt}f(S_T)Z_T$ equals the deterministically discounted expectation $\mathbb{E}e^{-rt}f(S_T^*)$, where the expectation is with respect to the **drift modified** geometric Brownian motion satisfying the SDE

$$\frac{dS_t^*}{S_t^*} = \bar{r}dt + \Sigma d\bar{B}_t$$

b) The value $V_t = \mathbb{E}e^{-r(T-t)}[f(S_T^*)/F_t]$ at time t on a claim $f(S_T)$ in a GBM market satisfies the **Black Scoles PDE**

$$\frac{\partial}{\partial t}V + \frac{\sigma^2 S^2}{2}\frac{\partial^2}{\partial S^2}V + rS\frac{\partial}{\partial S}V + rV = 0$$

with final condition $V(T, S_T) = f(S_T)$.

c) **The "risk neutral" valuation formula.** When $r = 0$ the GBM S_t is a martingale under the modified discounted measure.(For general r S_t is a "r martingale" under the modified discounted measure, which means that $e^{-rt}S_t$ is a martingale (see note 3) below). In conclusion, the valuation formula may be written as

$$v_t = \mathbb{E}^*[e^{-rt}f(S_T)/F_t]$$

where $\mathbb{E}^*$ denotes expectation extra discounted (weighted) by Z_t, under which the asset price is a martingale.

Notes:

- 1) Theorems 6.1 and 6.2 are the basis of the "risk neutral" approach to the valuation of derivatives for GBM markets. While the way to derive them is quite involved, there is an interesting simplification in the final answer described in Theorem 6.2 b); namely, it is unnecessary to estimate the expected rate of return μ of assets and to compute their market prices of risk θ, since these parameters do not appear in the Black Scholes P.D.E.

- 2) Even more significant is the fact that the final "risk neutral" valuation formula given in Theorem 6.2 c) does not involve explicitly σ any,ore, but maybe expressed instead in terms of the process S_t alone, **without any reference to a parametric model**. Indeed, while the "risk neutral" valuation formula was first discovered in the context of GBM markets, it turned out to be relevant under **any probabilistic assumptions** for the market, as was first discovered in a seminal paper of Harrison and Pliska.

  What is different in other models is that the martingale measure stops being unique. If for example the rates of return are modeled in discrete time by the so called multinomial model under which $\frac{dS_t^*}{dS_t^*}$ may take a finitely number of values r_w any distribution p_w satisfying the "balancing" (risk neutral) constraint

$$\sum p_w r_w = 0$$

  may be used the pricing problem, if our only demand is to preclude "arbitrage opportunities". In fact, as we will show in the next section, the seller and the buyer of an option are bound to disagree on which risk neutral measure to use.

  The main point is that the while there may be many martingale= risk neutral= =balancing measures, pricing will always involve choosing one of these measures. This came to be known as the **Risk neutral valuation** principle. Furthemore, the constraints $\sum p_w r_w = 0$ defining the set of risk neutral measures do not require statistical estimation of the evolution of asset prices, which is quite difficcult in nature. Incorporating statistical information in the pricing of derivatives is doubtlessly a worthy challenge, which has only started to be met.

- 3)The pricing measure and the Black Scoles PDE are completely independent of the expected rate of return μ (or the growth rate g of an asset). We have the same price for options on all the stocks with the same volatility σ, no matter if the expected rate of return is up or down! This is a somewhat unfortunate consequence of various oversimplifying assumptions of this model:

  "Short" cash positions and long cash positions are assumed to have the same interest rates and "short" cash positions and long cash positions are assumed to have the same liquidity. Thus, a stock which goes spectacularly down is not worse than one that goes up, since it is possible to sell it "short" with no particular penalty. The Black Scholes value of an option is only proportional to the volatility of the stock, since this measures the amount of trading which is necessary for replicating. However, since it is assumed that there are no transaction costs, the volatility does not reflect accurately the trading costs. A correction of the volatility which incorporates trading costs has been suggested by Leland. Another approach of Morton and Pliska introduces a no trading zone around the optimal hedging position; upon reaching the boundaries of this zone, the position is readjusted to the optimal hedging position. This prevents "replicating" strategies involving continuous trading.

- 4) In the presence of nonzero interest rates r, the definition of a martingale has to be slightly modified. A process $X(t)$ is an r martingale if $e^{-r\,t}X(t)$ is a usual martingale (thus, the expectation of an r martingale increases like $e^{r\,t}$). Mathematically, this is a trivial generalization since if we measure an r martingale with respect to an artificial currency which increases as $\exp(rt)$, it becomes an usual martingale. This cancels the effect of interest rates and puts us effectively in the case $r = 0$. Changing back to the actual currency is of course an easy multiplication by $e^{r\,t}$. Thus, depending on taste, we can either work with r martingales, or keep r out of our formulas and work with usual martingales, but then change to the actual currency at the end.

The martingale approach is especially efficient in the case of **complete markets,** which are markets for which the martingale pricing measure is unique. For them, an exact analog of the GBM result holds.

Theorem 4.3 (**Fundamental theorem of derivative pricing**

a) In a "complete market" with a unique "risk neutral" martingale pricing measure denoted by $\mathbb{E}^*$ the initial value of a replicating portfolio of any future claim $H_T = f(S_T)$ has to be $v_0 = e^{-r\,T}\mathbb{E}^* H_T$

b) The value of the replicating portfolio at any time $t < T$ has to be $V_t = V(t, S_t) = e^{-r\,(T-t)}\mathbb{E}^*[H_T/F_t]$

c) The replicating portfolio has to contain at time t $\Delta_t = \frac{\partial V_t}{\partial S_t}$ units of stock.

Exercise Suppsing that $r = 0$ compute the value at any time t of a forward with final payoff S_T and find the optimal hedge.

Solution: Using the previous theorem we have: $V(t, S_t) = E^*[S_T/F_t] = S_t$ (by the martingale property of the risk neutral measure).The optimal hedge $\Delta_t = \frac{\partial S_t}{\partial S_t}$ is is to always keep one unit of stock.

4.5 ** Hedging strategies for call options

The independence of the Black Scholes PDE from the estimated expected rate of return is so counterintuitive that it warrants a more detailed examination. We will attempt to explain it by looking more closely at the hedging of call options. Without loss of generality we may discuss only the case $r = 0$.

The simplest possible hedging coming to mind is the naive "stop loss" policy of keeping the stock (and the loan) when the price S_t is above K and liquidating them when it gets below. This would result in an initial value and price of $(S_0 - K)_+$. A suspicious thing about this strategy is that "out of money" call options would have 0 price. The astute buyer would then get a lot (zillions)! Since one of a zillion options is bound to get "in the money", the astute buyer would realize a profit for nothing (an "arbitrage").

In discrete time however the stop loss strategy can not work, since whenever you try to sell the stock when it moves below K, or when you try to buy it again when it moves above K you are bound to lose a bit.

The stop loss would be however the right strategy in a "smooth" continuous time market in which $\sigma = 0$ and thus there is no Brownian motion. There is apparently something terribly wrong about the assumption of a smooth continuous time stock market (in which people can get rich by buying 0 cost products).

The "bull" and "bear" bounds Let us note that sure knowledge that the stock will end up above K (that the option ends "in the money") would imply a price of $(S_0 - K)_+$ for the option (since the hedger can take a loan and buy the stock, being sure that the loan will be repaid). It turns out that any uncertainty about the future increases the value of an option, i.e. under any probabilistic model $(S_0 - K)_+ \leq v_0$

On the other hand, in an extremely uncertain market the hedger might be forced to get the stock upfront, without taking any loan (since all the money put in acquiring the stock might well be lost). This strategy wpould lead to a price of S_t and it turns out that the value of an option, should never be larger than this, i.e. under any probabilistic model $v_0 \leq S_0$

We will call these bounds the **"bull" and "bear" bounds**.

$$(S_0 - K)_+ \leq v_0 \leq S_0$$

It may be easily checked that they are also the particular cases of the Black Scholes correspond to $\sigma = 0$ (no uncertainty) and $\sigma = \infty$ (unbounded uncertainty).

The Black Scholes hedge The stop loss strategy always keeps $\Delta = 1$ or $\Delta = 0$ units of stock. The Black Scholes hedge (for $\sigma \neq 0$) on the other hand recommends keeping always some fractions between 0 and 1 of the stock. This fraction changes smoothly in time and converge to 0 or 1 at expiration as appropriate. This maybe viewed as an attempt to preempt the large expenses incurred by the stop loss strategy when it overshoots.

The value of the hedging portfolio to be kept is increasing when the "volatity" parameter σ increases, and this parallels the fact that in volatile markets hedging is more expensive.

It is probably this simple quantifying in one mathematical parameter σ of the obvious difficulties experienced in volatile markets which explains the success of the Black Scholes formula, despite the fact that there is clear evidence that this model does not fit observations of asset prices. In answer to this inconsistency we will examine in a later section on **stochastic volatility** a generalization of the Black Scholes model in which it is assumed that the volatility σ itself is unknown, being modeled by some stochastic process.

4.6 ** Perfect Replication with the Black Scholes portfolio

Surprisingly, while perfect replication is not possible in multinomial models, it becomes again possible (in an asymptotic sense) for geometric Brownian motion (even though its increments may take an infinite number of values)! For a heuristic explanation, consider a claim $f(S_t)$ which we would like to hedge over an infitesimal period of length h. What should we charge and how should we hedge?

A first approxmation of the answer is provided by Taylor's formula:

$f(S_h) \approx f(S_0) + f'(S_0)dS_h + ...$

The most likely candidate for the initial cost to charge is $f(S_0)$. But how can we guard against a possible large change in value $f'(S_0)dS_h + ...$? (Recall that the price change dS_h is assumed to have large variations!)

Note that if we are holding a portfolio $\varphi S_t + \psi$, the capital gains are precisely φdS_h. So, if we choose $\varphi = f'(S_0)$, we will succeed to cover the major part of the change in value (the dominant term in its Taylor expansion) by the capital gains.

We arrive thus at the "Δ" hedging rule: For short periods of time, keep in the portfolio a number $\Delta = f'(S)$ of stock units equal to the derivative of the claim with respect to the price.

Example 1: For a call put near expiration we should hold one unit of stock if "in the money" and none else.

Example 2: For a claim which pays S_T^2 we should hold $2S_{T-h}$ units of stock for our final hedge.

The point to emphasize here is that by choosing correctly the one decision variable φ we can reproduce approximately the change in value of an arbitrary claim.

To extend the "Δ" hedging rule over longer intervals of time, it is necessary first to specify what value needs to be kept in a hedging portfolio at time t.

In his Noble prize winning work, Merton showed that this value has to be of the form $v(S_t, t)$ (i.e. a function of the current price S_t and the remaining time to expiration) and that it has to verify the partial differential equation:

$$v_t + \frac{\sigma^2 S^2 v_{SS}}{2}$$
$$v(T) = f(S_T, T)$$

where $f(S_T, T)$ maybe the arbitrary payoff of any European claim.

If in addition one hedges by continuously rebalancing the portfolio so that it always contains $\varphi_t = f_S(S_t, t)$ units of stock (and a loan $\psi_t = v(S_t, t) - S_t f_S(S_t, t)$), then replication is perfect.

More precisely, if the price evolution follows a geometric Brownian motion and if we were to trade after very small intervals of time h, the total of the replication errors involved would converge to 0 in the limit $\phi \to 0$. The argument involves using Ito's formula and is roughly reproduced below.

Definition: A trading strategy in discrete time is a sequence of positions φ_i, ψ_i in stock and bond respectively, taken at time i and maintained until time $i+1$.

The value of the associated portfolio at time i is thus $V_i = \varphi_i S_i + \psi_i$.

Definition: A portfolio in discrete time (when $r = 0$) is called **selffinancing** if the new position taken at time $i + 1$ involves no additional expenses, i.e. if:

$\varphi_{i+1} S_{i+1} + \psi_{i+1} = V_{i+1} = \varphi_i S_{i+1} + \psi_i$

It is easy to obtain by eliminating ψ_i the "capital gains" equation: $V_{i+1} = V_i + \varphi_i(S_{i+1} - S_i)$, which further implies that the value at time n is given by the sum of the initial investment v_0 and the total capital gains reaped:

$$V_n = v_0 + \sum_{i=0}^{n-1} \varphi_i (S_{i+1} - S_i) \tag{66}$$

The sequence φ_i can be completely arbitrary, since any stock position may be funded by taking an appropriate loan ψ_i. It is thus convenient to eliminate ψ_i from our discussion, and to redefine a selffinancing portfolio as a portfolio satisfying the "capital gains" equation (66).

Analogously, in continuous time a portfolio is called **selffinancing** if its value may be represented as a (Ito) stochastic integral with respect to the stock price

$$V_T = V_O + \int_0^T \varphi_t dS_t.$$

Merton's argument was roughly as follows: let V_t denote the "fair price" of a claim H at some time $t < T$ prior to expiration, and let φ_t, ψ_t denote the "best" trading strategy, where "fair" and "best" are meanwhile taken informally to mean that we will strive to minimize replication errors.

The key point is to realize that the "fair value=price" at time t should be some function of the current stock price S_t and of the current time (or the remaining time until expiration $\tau = T - t$). Thus, $V_t = v(S_t, t)$.

Let us suppose that we have managed to hedge exactly until time t, i.e. we have found a selfinancing strategy whose current worth is $V_t = v(S_t, t)$. Our next step should be to hedge the next change of the value function, which by Ito's formula is:

$V_{t+h} - V_t = v_S dS_t + (\frac{v_{SS}}{2}[dS_t]^2 + v_t dt$

On the other hand, the value of the change of a selffinancing portfolio at time $t+h$ should be

$V_{t+h} - V_t = \varphi_t dS_t.$

It follows that the hedging strategy should be to hold $\varphi = v_S$ units of stock (this determines automatically the ψ_t by $\psi_t = v - Sv_S$). If the replication error is going to be 0, we must also ensure that the second part of the change in the value function is 0, i.e.

$$\sigma^2 S^2 \frac{v_{SS}}{2} + v_t = 0 \tag{67}$$

Thus, the value at time t must satisfy the Black-Scholes partial differential equation (67).

The partial differential equation is the same for any European claim. Ihe final payoff provides a boundary condition. For example in the case of the call option we must have $v(S_T, T) = (S_T - K)_+$ By solving the equation together with this boundary condition one may recover the famous Black Scholes formula (which we first derived as an expectation).

An interesting question is what happens in a continuous market where the stock follows geometric Brownian motion with parameters μ, σ, if one hedges via Black Scholes with a wrong parameter σ'.

Theorem 2: Hedging "discretely" via Black Scholes with a wrong parameter σ' in a market where the stock follows geometric Brownian motion with parameters (μ, σ) will result in the limit in hedging costs given by:

$$(\frac{\sigma^2}{\sigma'} - \sigma') \int_0^T \frac{S_t \varphi(s_t)}{2\sqrt{T-t}} \, dt.$$

Notes: 1) The limiting hedging cost is 0 iff $\sigma' = \sigma$. In this case we call the hedging strategy "self financing".

2)The Black Scholes hedging strategy is independent of μ; as long as we can estimate σ, we can attain 0 hedging costs by trading continuously in a way which disregrads altogether the stocks long term "prospects"! In continuous trading, we are only concerned with the short term features (volatility) of the stock!

Exercise 1: Write a computer program (in maple?) $sim(\mu, \sigma, N)$ which produces a series of N observations $S(0), S(1), S(2), ..., S(N)$ from a geometric Brownian motion starting at $S_t = \frac{dS_t}{S_t} = \mu dt + \sigma dB_t$ with parameters g, σ^2. (Hint: Obtain the discretized sample of the Brownian motion with drift by adding normal random variables N_{g,σ^2} and then exponentiate it. Investigate whether the following theoretical results may be observed by simulation:

a) $\lim_{n\to\infty} S_{-1,1}(N) = 0$.

b) $\lim_{n\to\infty} S_{-.5,1}(N)$ does not exist.

c) $\lim_{n\to\infty} \frac{S_{0,1}(N)}{e^{\frac{N}{2}}} = X$ where X is a random variable with expectation 1.

Exercise 2: Write a computer program (in maple?) to investigate the performance of Black Scholes hedging $hed(K, N, \sigma', \sigma, \mu)$ of call options with starting price 1, exercise price K, 0 interest rate, expiration date N, and assumed volatility σ', when the real parameters are σ, μ. Recall that the Black Scholes hedging costs are given by $hed(K, N, \sigma', \sigma, \mu) = \sum_{i=0}^{N-1} (V_{i+1} - V_i) - \Delta_i (S_{i+1} - S_i)$ where $V_i = \Phi(s_i) - K\Phi(l_i)$ is the recommended Black Scholes value of the hedging portfolio and $\Delta_i = \frac{\partial V}{S} = \Phi(s_i)$ is the hedge (we put $s, l = \frac{\log \frac{S_0}{K} \pm \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}$).

Note: Both the Black Scholes value and the initial hedge are available in the maple finance package with(finance) (try help("blackscholes") in order to see how to use the command blackscholes). The last parameter "hedge" of the command blackscholes(amount, exercise, rate, nperiods, sdev , hedge) is expected to be an unassigned variable (for outputting the hedge).

a) Investigate the magnitude of the hedging costs on a series with real volatility σ (produced by calling the function $sim(a, \mu, \sigma, N)$ implemented before to simulate stock price evolution). For example, try $N = 100, K = 1.2, \sigma = 1, \mu = 0$ and plot the magnitude of the hedging costs as a function of σ' for $k = 20$ values ranging from 0.0001 to $2\sigma + 0.0001$.

Repeat for a couple of other values of K.

b) Investigate also the performance when μ varies by calling $hed(K, N, \sigma, \sigma, \mu)$ for the same values of K, N, σ as before and with the correct $\sigma' = \sigma$, for a range of 10 values of μ starting from $\frac{-\sigma^2}{2}$ to $\frac{-\sigma^2}{2}$.

4.7 Exchange options

It is possible to extend the Black Scholes theory to the case of an **exchange option** with final value $(S_T^1 - S_T^2)_+$ where the two assets are modeled by $S_t^1 = S_0^1 e^{g^1 t + \sigma^1 B_t^1}$ and $S_t^2 = S_0^2 e^{g^2 t + \sigma^2 B_t^2}$, where B^1, B^2 are two Brownian motions with correlation ρ, by using the price of the second asset as an artificial currency (called "numeraire"). Clearly, the "numeraire" price of the second asset is equal to 1 at any time t and thus using "numeraire" puts us in the previously discussed situation when the second asset is constant. This is however precisely the case of a call option in a market with 0 interest rate, for which we may apply the classical Black Scholes formula.

The "numeraire" price of the first asset becomes $Y_t = \frac{S_t^1}{S_t^2}$

Exercise 1 a) What is the distribution of Y_t?

b) Find the "numeraire" Black Scholes value for exchanging $Y(T)$ by 1, as well as the value of the exchange option $\mathbb{E}(S_T^1 - S_T^2)_+$ measured in original currency. [3]

c) Find the price of the standard call and put options on a GBM asset in a market with interest rate r and show that they satisfy the "call-put" parity formula.

Solution

a) Y_t is also a geometric Brownian motion with parameters $g = g_1 - g_2$ and with variance $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$.

b) The "fair numeraire price" of the exchange option is obtained by plugging $Y_0 = \frac{S_0^1}{S_0^2}$ instead of S_0 and 1 instead of K in the $r = 0$ Black Scholes formula, yielding:

$$\frac{S_0^1}{S_0^2}\Phi\left(\frac{ln(\frac{S_0^1}{S_0^2}) + \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}\right) - \Phi\left(\frac{ln(\frac{S_0^1}{S_0^2}) - \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}\right).$$

The "fair original currency price" at time t is obtained by multiplying with S_t^2. Thus, for the initial value we have to multiply with S_0^2, yielding $S_0^1\Phi(s) - S_0^2\Phi(l)$ where we put $s,l = \frac{\frac{S_0^1}{S_0^2} \pm \frac{\sigma^2 T}{2}}{\sqrt{\sigma^2 T}}$.

c) In the case of the call (put) options in markets with non zero interest rate we have $S_t^2 = S_0^2 e^{rt}$ and the final value is K. The initial value is thus $S_0^2 = Ke^{-rT}$. We get the respective formulas

$$C = S_0\Phi(s) - Ke^{-rT}\Phi(l) \qquad\qquad P = Ke^{-rT}\Phi(-l) - S_0\Phi(-s)$$

Since $\Phi(-x) = 1 - \Phi(x)$ we get the "put-call" parity relation: $P_0 = Ke^{-rT} - S_0 + C_0$ which has the clear investment interpretation that buying the stock and a put and taking a loan is the same as buying a call.

Exercise 2 (**valuation of puts**) Let S_t be a geometric Brownian motion with volatility σ and growth parameter g. The risk free interest rate is r.

Find the probability that the option will be exercised in the real world and in a "risk neutral world" (where the stock price moves according to the equivalent martingale measure).