Russian and American put options under exponential phase–type Lévy models

Søren Asmussen [a], Florin Avram [b], Martijn R. Pistorius [c,*]

[a] Department of Mathematical Sciences, Aarhus University, Ny Munkegade, 8000 Aarhus C, Denmark

[b] Département de Mathématiques LMA, Université de Pau, 64000 Pau, France

[c] Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK

Abstract

Consider the American put and Russian option [46,47,22] with the stock price modeled as an exponential Lévy process. We find an explicit expression for the price in the dense class of Lévy processes with phase–type jumps in both directions. The solution rests on the reduction to the first passage time problem for (reflected) Lévy processes and on an explicit solution of the latter in the phase–type case via martingale stopping and Wiener-Hopf factorisation. The same type of approach is also applied to the more general class of regime switching Lévy processes with phase-type jumps.

Key words: Lévy process, Markov additive process, first passage time, Wald martingale, Wiener-Hopf factorisation, Russian option, optimal stopping.
1991 MSC: 60G40, 90A09

---

1 Introduction

Consider a model of a financial market with two assets, a savings account with value $B = \{B_t\}_{t\geq 0}$ and an asset with price process $S = \{S_t\}_{t\geq 0}$. The evolution

* Corresponding author.
Email addresses: asmus@imf.au.dk (Søren Asmussen), Florin.Avram@univ-pau.fr (Florin Avram), pistoriu@mth.kcl.ac.uk (Martijn R. Pistorius).

of B is deterministic, with

$$B_t = \exp(rt), \qquad r > 0, t \geq 0,$$

and the asset price is random and evolves according to the exponential model

$$S_t = S_0 \exp(X_t), \qquad S_0 = \exp(x), \qquad t \geq 0,$$

where $X = \{X_t\}_{t\geq 0}$ is some Lévy process. If X has no jumps, it can be represented by $X_t = \sigma W_t + \mu t$, with $x, \mu, \sigma \in \mathbf{R}$ and $W = \{W_t\}_{t\geq 0}$ a standard Wiener process; this is the classical Black-Scholes model. There has been considerable interest in replacing the classical Black-Scholes model by exponential Lévy models allowing also for jumps. This development is motivated by superior fits to the data and hence improved pricing formulas and hedging strategies, as well as by theoretical considerations outlined in [27].

The search for a special Lévy model to outperform the Black-Scholes model was initiated by Mandelbrot [38,39] and Fama [25,26] followed by Merton, with the jump-diffusion with Gaussian jumps, and continues nowadays in the work of Carr, Chang, Madan, Geman and Yor who propose the variance-gamma model [37,18], of Eberlein who proposes the hyperbolic model [23], of Barndorff-Nielsen with the normal inverse Gaussian model [11], of Kou who proposed a jump-diffusion with exponential jumps [34] and of Koponen who introduced the Koponen family [35], which was later extended (e.g. [15,20]). There are still many statistical issues which will need to be resolved before an appropriate replacement of the Black Scholes model can emerge. Our paper adresses only the issue of the analytical tractability of pricing certain perpetual American type options. We propose a jump-diffusion model where the jumps form a compound Poisson process with jump distribution of *phase–type* (e.g. [43,5,6], see further Section 2). On the one hand this *phase–type* model is rich enough, since this class of processes is known to be dense in the class of all Lévy processes, and on the other hand for many options the model is analytically tractable.

We illustrate this in the case of the American put option and the Russian option. The last one was originally introduced by Shepp and Shiryaev in the context of the Black-Scholes model [22,46,47,29,36]. The pricing of the Russian option rests on a well known reduction to the *first passage time problem* for a Lévy process reflected at its supremum, making it somewhat more difficult than the analogous problem for the unconstrained Lévy process (which is used to solve the pricing problem for barrier and perpetual American options). We note that special solutions of this problem – see [10] and [42] – are currently available only under spectrally one sided Lévy models. The purpose of our note is to draw attention to the fact that under the phase–type assumption, easily implementable solutions for both the unconstrained and the reflected first passage time problems exist as well for *spectrally two sided* Lévy processes (and hence for the pricing of perpetual American put and Russian options). In fact, we show that the method employed – of obtaining barrier crossing probabilities via a martingale stopping approach – works equally for barrier problems under the much more general class of *regime switching* exponential Lévy models with phase–type jumps, or for the regime switching Brownian motion recommended for example by Guo [30]. Their analytical tractability suggests that this potentially very flexible class of models (which depart from the unrealistic assumption of independent increments of the Lévy models) deserves to be more fully investigated.

The rest of the paper is organised as follows. Section 2 presents the model, the problem and its reduction to the first passage time problems for (reflected) Lévy processes. The martingale stopping approach for reflected and nonreflected Lévy processes is reviewed in Section 3, including explicit formulae for the pricing of the perpetual American put option and the Russian option. Finally, the solution of the first passage problem for reflected regime switching phase–type Lévy models via an embedding into a regime switching Brownian motion is presented in Section 4. Most proofs are relegated to Section 5.

2 Model and problem

We introduce now the model we consider.

2.1 Phase-type distributions

A distribution F on $(0,\infty)$ is *phase-type* if it is the distribution of the absorbtion time ζ in a finite state continuous time Markov process $J=\{J_t\}_{t\geq 0}$ with one state Δ absorbing and the remaining ones $1,\dots,m$ transient. That is, $F(t)=\mathbb{P}(\zeta\leq t)$ where $\zeta=\inf\{s>0:\ J_s=\Delta\}$. The parameters are m, the restriction $\boldsymbol{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\boldsymbol{\alpha}=(\alpha_1\ \dots\ \alpha_m)$ where $\alpha_i=\mathbb{P}(J_0=i)$. For any $i=1,\dots,m$, let t_i be the intensity of a transition $i\to\Delta$ and write $\boldsymbol{t}=(t_1\ \dots\ t_m)'$ for the (column) vector of such intensities. Note that $\boldsymbol{t}=-\boldsymbol{T}\boldsymbol{1}$, where $\boldsymbol{1}$ denotes a column vector of ones. It follows that the cumulative distribution F is given by:

$$1-F(x)=\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{1}, \tag{1}$$

the density is $f(x)=\boldsymbol{\alpha}\mathrm{e}^{\boldsymbol{T}x}\boldsymbol{t}$ and the Laplace transform is given by $\hat{F}[s]=\int_0^\infty \mathrm{e}^{-sx}F(\mathrm{d}x)=\boldsymbol{\alpha}(s\boldsymbol{I}-\boldsymbol{T})^{-1}\boldsymbol{t}$. Note that $\hat{F}[s]$ can be extended to the complex plane except at a finite number of poles (the eigenvalues of $\boldsymbol{T}$). A representation of the form (1) for the distribution function F is called *minimal* if there exists no number $k < m$, k-vector $\boldsymbol{b}$ and $k \times k$-matrix $\boldsymbol{G}$ such that $1 - F(x) = \boldsymbol{b}\mathrm{e}^{\boldsymbol{G}x}\mathbf{1}$.

Phase-type distributions include and generalize exponential distributions in series and/or parallel and form a dense class in the set of all distributions on $(0, \infty)$. They have found numerous applications in applied probability, see for example [5,6] for surveys. Much of the applicability of the class comes from the probabilistic interpretation, in particular the fact that that the overshoot distributions $F(x+y)/(1-F(x))$ belong to a finite vector space. More precisely, the overshoot distribution is again phase–type with the same m and $\boldsymbol{T}$ but α_i replaced by $\mathbb{P}(J_x = i | \zeta > x)$, which is reminiscent of the memoryless property of the exponential distribution ($m = 1$) and explains the availability of many matrix formulas which generalize the scalar exponential case.

2.2 Lévy phase-type models

Let $X = \{X_t\}_{t \geq 0}$ be a Lévy process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, a stochastic basis that satisfies the usual conditions. We consider X which can be represented as follows

$$X_t \; = \mu t + \sigma W_t + \sum_{k=1}^{N^{(+)}(t)} U_k^{(+)} - \sum_{\ell=1}^{N^{(-)}(t)} U_\ell^{(-)}, \tag{2}$$

where W is standard Brownian motion, $N^{(\pm)}$ are Poisson processes with rates of arrival $\lambda^{(\pm)}$ and $U^{(\pm)}$ are i.i.d. random variables with respective jump size distributions $F^{(\pm)}$ of phase–type with parameters $m^{(\pm)}, \boldsymbol{T}^{(\pm)}, \boldsymbol{\alpha}^{(\pm)}$. All processses are assumed to be independent. Equivalently, for $s \in \mathrm{i}\mathbf{R}$, the Lévy exponent κ of X, defined by $\kappa(s) = \log \mathbb{E}[\exp(sX_1)]$, is

$$\kappa(s) \;=\; s\mu + s^2\frac{\sigma^2}{2} + \lambda^{(+)}\Big(\hat{F}^{(+)}[-s] - 1\Big) + \lambda^{(-)}\Big(\hat{F}^{(-)}[s] - 1\Big), \tag{3}$$

where $\hat{F}^{(\pm)}[s] = \boldsymbol{\alpha}^{(\pm)}(s\boldsymbol{I} - \boldsymbol{T}^{(\pm)})^{-1}\boldsymbol{t}^{(\pm)}$. As above, $\kappa(s)$ can be extended to the complex plane except a finite number of poles (the eigenvalues of $\boldsymbol{T}^{(\pm)}$); this extension will also be denoted by κ. To avoid trivialities, in the sequel we will exclude the case that X has monotone paths.

Any Lévy process may be approximated arbitrarily closely by processes of the form (2):

Proposition 1 *For any Lévy process X, there exists a sequence $X(n)$ of Lévy processes of the form (2) such that $X(n) \to X$ in $D[0, \infty)$.*

Proof Let d be the Prokhorov distance on the space D of right-continuous functions with left-limits from $[0, \infty)$ to $\mathbf{R}$ (see e.g. Chapter VI of [31]). Choose first $X'(n)$ as an independent sum of a linear drift, a Brownian component and a compound Poisson process such that $d(X, X'(n)) \leq 1/n$. Use next the denseness of phase–type distributions to find $X(n)$ of the form (2) with $d(X(n), X'(n)) \leq 1/n$. QED

Remark 1 The approximation in Proposition 1 is easy to carry out in practice: the compound approximation is obtained by just restricting the Lévy measure to $\{|x| > \epsilon\}$, and to get to phase–type jumps, the relevant methodology for fitting a phase–type distributions to a given distribution (or a set of data) is developed in [4] for traditional maximum likelihood and in [14] in a Bayesian setting.

In complete markets (with a unique risk-neutral martingale measure $\mathbb{P}^*$ under which $\mathbb{E}^*[\exp(X_t)] = \exp(rt)$ where r is the riskless discount rate), arbitrage free pricing is equivalent to computing expectations under this measure $\mathbb{P}^*$. Under the Lévy model (2) with non-zero jump component however, the market is incomplete, i.e. not all claims can be hedged against. In this case there are infinitely many equivalent martingale measures, and some choice must be made. We use here the so called Cramér-Esscher transform or exponential tilting proposed by Gerber and Shiu [28], which preserves the Lévy structure, and as shown in Chan [19], is indeed the solution to some of the most common criteria for selecting an equivalent martingale measure. Note that the Esscher transform preserves the phase-type structure of the log-price X (see Appendix A). From now on we assume that we are working *under the chosen equivalent martingale measure*. That is, we assume that the Lévy exponent κ satisfies under $\mathbb{P}$

$$\kappa(1) = \log \mathbb{E}[\exp(X_1)] = r, \qquad \text{(EMM)}$$

Remark 2 Many of the computations involving Lévy processes are based on finding the roots of the "Cramér-Lundberg equation" (see [6] for terminology)

$$\kappa(s) = a \qquad (4)$$

(for some a). From this perspective, working under the equivalent martingale measure means $s = 1$ is one of the roots of this equation when $a = r$.

Remark 3 Using Appendix A, we can easily convert parameters of X under the real world measure into parameters under the Esscher transform and vice versa.

2.3 American put option

The a-discounted perpetual American put option with strike K gives the holder the right to exercise at any $\{\mathcal{F}_t\}$-stopping time τ yielding the pay-out

$$e^{-a\tau}(K - S_\tau)^+, \qquad a \geq 0, \tag{5}$$

where $c^+ = \max\{c, 0\}$. Recall that the process X satisfies (EMM). Then the arbitrage-free price corresponding to the chosen martingale measure is given by

$$U^*(x) = \sup_\tau \mathbb{E}_x[e^{-(r+a)\tau}(K - S_\tau)^+] \tag{6}$$

where the supremum runs over all $\{\mathcal{F}_t\}$-stopping times τ, $\mathbb{E}_x$ denotes the expectation with respect to the measure $\mathbb{P}_x$ under which $\log S_0 = X_0 = x$. Let $I_\delta = \inf_{0\leq t\leq \eta(\delta)} X_t$ denote the infimum of X up to $\eta(\delta)$, an indepedent exponential random variable with parameter $\delta = r + a$. Mordecki [40] has shown that, for a general Lévy process X,

$$U^*(x) = \mathbb{E}_x[e^{-\delta T^*}(K - e^{X_{T^*}})],$$

where the optimal stopping time T^* is given by the first passage time of the process X below the level k^*,

$$T^* = T(k^*) = \inf\{t \geq 0 : X_t \leq k^*\}, \tag{7}$$

where $\exp(k^*) = K\ \mathbb{E}[e^{I_\delta}]$. (cf. Darling et al. [21] for the solution of a similar optimal stopping problem in discrete time).

2.4 The Russian option

The Russian option is an American type option which gives the holder the right to exercise at any almost surely finite $\{\mathcal{F}_t\}$-stopping time τ yielding payouts

$$e^{-a\tau} \max\left\{M_0, \sup_{0\leq u\leq\tau} S_u\right\}, \qquad M_0 \geq S_0, a > 0.$$

The constant M_0 can be viewed as representing the "starting maximum" of the stock price (say, the maximum over some previous period $(-t_0, 0]$). The positive discount factor a is necessary in the perpetual version to guarantee that it is optimal to stop in an almost surely finite time and the value is finite. Since X satisfies (EMM), the arbitrage-free price of the Russian option for this martingale measure is given by

$$V^*(x, m) = \sup_\tau \mathbb{E}_x\left[e^{-(r+a)\tau} \max\left\{e^m, \sup_{0\leq u\leq\tau} S_u\right\}\right], \tag{8}$$

where the supremum is taken over the set $\mathcal{T}$ of all almost surely finite $\{\mathcal{F}_t\}$-measurable stopping times, $m = \log(M_0)$ and $\mathbb{E}_x$ denotes expectation with the initial condition $X_0 = \log(S_0) = x$. Let $\overline{X}_t = \max\{\sup_{s\leq t} X_s, m\}$ denote the supremum of the Lévy process and write $Y_t = \overline{X}_t - X_t$ for the process reflected at its supremum level (starting at $Y_0 = m - X_0$). The key simplification discovered by Shepp and Shiryaev (for the standard Black-Scholes model) is that the optimal stopping time τ^* is of the form

$$\tau^* = \tau(k^*) = \inf\{t \geq 0 : Y_t \geq k^*\}, \tag{9}$$

i.e. the first time when the *reflected process* Y upcrosses a certain *constant* (positive) exercise level k^* (which may be found by solving a one dimensional optimization problem). If X is a general Lévy process, Theorem 1 below states that the optimal stopping time is still of the form (9).

Theorem 1 *Let X be a general Lévy process which satisfies (EMM). Then the value function $V^*(x,m)$ of the* two dimensional stopping problem *(8) is given by:*

$$V^*(x,m) = \mathrm{e}^x v^*(m-x), \tag{10}$$

where $v^*(m-x)$ is the solution of the one dimensional stopping problem *of finding a function v^* and a $\tau^* \in \mathcal{T}$ such that*

$$v^*(y) = \sup_{\tau\in\mathcal{T}} \mathbb{E}^{(1)}_y[\mathrm{e}^{-a\tau+Y_\tau}] = \mathbb{E}^{(1)}_y[\mathrm{e}^{-a\tau^*+Y_{\tau^*}}], \tag{11}$$

where $\mathbb{P}^{(1)}_y$ denotes the "tilted" probability measure given on $\mathcal{F}_t$ by $\mathrm{d}\mathbb{P}^{(1)}|_{\mathcal{F}_t} = \exp(X_t - x - \kappa(1)t)\mathrm{d}\mathbb{P}_x|_{\mathcal{F}_t}$, with $Y_0 = y$. The function v^* is convex and the optimal stopping time τ^* is the same in both problems, i.e. $\tau^* = \tau(k^*)$ with k^* given by

$$k^* = \arg\max_{k\geq 0}[\mathrm{e}^k v^{(1)}_k(0)]$$

where $v^{(1)}_k(y) = \mathbb{E}^{(1)}_y[\mathrm{e}^{-a\tau(k)+Y_{\tau(k)}-k}]$. Moreover, $k^* = \arg\max_{k\geq y}[\mathrm{e}^k v^{(1)}_k(y)]$ for all $y > 0$.

In Section 5 we provide the proof. The proof draws on the experience of [46,47,36] and uses standard optimal stopping theory. In Section 3.3.1, an explicit expression is given for the optimal level k^* if X is of the form (2).

To explicitly solve our problem of pricing the American put and the Russian option driven by phase–type Lévy processes X, the next goal will be the explicit evaluation of the first passage time functions of the process X at the stopping time (7) needed in (6) and of the process Y at the stopping time (9) required in (11). In the case of the American put, we will actually do the evaluation for the more general class of Lévy processes X that have as only restriction that the downward jumps are of phase–type. The evaluation may be achieved in principle by solving the corresponding Feynman-Kac integro-differential equation, which is tractable for this phase-type Lévy model and worked out in Section 5.3. In the next section, however, we will follow a different approach, exploiting the probabilistic interpretation of phase-type distributions and the fact that distributions of phase–type have a rational Laplace transform.

3 First passage time

In this section we first review the Wiener-Hopf decomposition and first passage time problem for the class of Lévy processes X with arbitrary positive jumps and negative jumps of phase–type. Results on Wiener-Hopf factorisations have appeared before in the literature at different places. Here we aim to develop a self-contained presentation illustrating our methods. Next we solve the first passage time process of the Lévy process reflected at its supremum for the smaller class of Lévy processes (2) with positive and negative jumps of phase–type. For background on passage problems for Markov chains, we refer to [33].

3.1 The Wiener-Hopf factorisation

We consider now $X = \{X_t\}_{t\geq 0}$ to be of the form

$$X_t = X_t^{(+)} - J_t^{(-)} \tag{12}$$

where $X^{(+)} = \{X_t^{(+)}\}_{t\geq 0}$ is a Lévy process without negative jumps and $J^{(-)} = \{J_t^{(-)}\}_{t\geq 0}$ is a compound Poisson process with intensity $\lambda^{(-)}$ and jumps of phase–type with parameters $(m^{(-)}, \boldsymbol{T}^{(-)}, \boldsymbol{\alpha}^{(-)})$. We assume that X has non-monotone paths. For s on the imaginary axis we denote by $\kappa(s) = \kappa_X(s) = \log \mathbb{E}[\exp(sX_1)]$ the Lévy exponent of X. By the jump-structure of X, κ can be analytically extended to the negative complex half-plan except finitely many poles, the eigenvalues of $\boldsymbol{T}^{(-)}$, and we will denote the analytic extension also by κ. Denote by $\mathcal{I}^{(-)} = \{i : \Re(\rho_i) < 0\}$ the set of roots ρ_i with negative real part of the Cramèr-Lundberg equation

$$\kappa(\rho) = \kappa_X(\rho) = a, \tag{13}$$

taken each as many times as its multiplicity.

Let now $M_a = \sup_{t\leq\eta(a)} X_t$ and $I_a = \inf_{t\leq\eta(a)} X_t$ be the supremum and infimum of X at an independent exponential random variable $\eta(a)$ with mean a^{-1}, respectively. Set for $\pm\Re(s) \geq 0$

$$\varphi_a^-(s) = \mathbb{E}[\exp(sI_a)], \qquad \varphi_a^+(s) = \mathbb{E}[\exp(sM_a)]. \tag{14}$$

The functions $s \mapsto \varphi^{\mp}(s)$ are analytic for s with $\pm\Re(s) > 0$, respectively. Note by a Tauberian theorem that $\varphi_a^-(\infty) = \mathbb{P}(I_a = 0)$ and $\varphi_a^+(-\infty) = \mathbb{P}(M_a = 0)$.

The moment generating functions $\varphi^{\mp}(s)$ of the supremum and infimum processes can be computed via the Wiener-Hopf factorisation, which states that for $a > 0$ we have:

$$a/(a - \kappa(s)) = \varphi_a^+(s)\varphi_a^-(s) \qquad \text{for all } s \text{ with } \Re(s) = 0$$

see e.g. [13, Thm. 1].

For Levy processes with phase-type jumps (2), a more explicit statement is possible, by identifying the singularities and zeroes of $a/(a - \kappa(s))$ with positive/negative real part (note that since $|\varphi_a^{\pm}(s)| \leq 1$ for s with $\Re(s) = 0$, there are no singularities with zero real part when $a > 0$).

We start with a statement of the Wiener-Hopf factorisation for the more general class of processes (12). Let $\mathcal{J}^{(-)} = \{i : \Re(\eta_i) < 0\}$ denote the set of roots of $a/(a - \kappa(\eta)) = 0$ with negative real part, taking again multiplicity into account. Note that if $i \in \mathcal{J}^{(-)}$, η_i is an eigenvalue of $\boldsymbol{T}^{(-)}$, although the converse need not be true by non-minimality of the phase–type representation (see e.g. [2, Ex. 4.1]).

Lemma 1 *Let X be a Lévy process of the form (12).*

(1) The distribution of $-I_a$ is a convex combination of an atom of size $\varphi_a^-(\infty)$ at zero and a phase–type distribution on $(0, \infty)$, with a number of phases equal to $\#\mathcal{I}^{(-)} = \#\mathcal{J}^{(-)}$ or $\#\mathcal{J}^{(-)} + 1$ according to whether $X^{(+)}$ is a subordinator or not. If the representation $(m^{(-)}, \boldsymbol{T}^{(-)}, \boldsymbol{\alpha}^{(-)})$ of $F^{(-)}$ is minimal, $\#\mathcal{J}^{(-)} = m^{(-)}$.

(2) The Wiener-Hopf factor φ_a^- is for $\Re(s) \geq 0$ given by

$$\varphi_a^-(s) = \frac{\prod_{j \in \mathcal{J}^{(-)}}(s - \eta_j)}{\prod_{j \in \mathcal{J}^{(-)}}(-\eta_j)} \cdot \frac{\prod_{i \in \mathcal{I}^{(-)}}(-\rho_i)}{\prod_{i \in \mathcal{I}^{(-)}}(s - \rho_i)}, \tag{15}$$

where the first factor is to be taken equal to 1 if X has no negative jumps.

(3) If the roots of (13) with negative real part are distinct,

$$\mathbb{P}(-I_a \in \mathrm{d}x) = \sum_{j \in \mathcal{I}^-} A_j^-(-\rho_j)\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad x > 0 \tag{16}$$

where $\boldsymbol{A}^- = (A_i^-, i \in \mathcal{I}^{(-)})$ are the partial fractions coefficients of the expansion:

$$\varphi_a^-(s) - \varphi_a^-(\infty) = \textstyle\sum_{i \in \mathcal{I}^-} A_i^- \rho_i(\rho_i - s)^{-1}, \tag{17}$$

where φ_a^- is interpreted in the sense of its analytical continuation.

Proof of Lemma 1 1. Following [2, p. 778], we embed first the process X killed at rate a into a regime-switching spectrally positive Lévy process $(J, \widetilde{X})$, by 'leveling out' the negative jumps (i.e. replacing them by linear pieces with slope -1). Here J is the finite state Markov process indicating the current phase of $\widetilde{X}$ (where the state is 0 if $\widetilde{X}$ is moving as $X^{(+)}$, Δ after the killing and otherwise the current phase of the underlying Markov process of the jump). Define the time-change $\gamma_t = \inf\{u \geq 0 : -\widetilde{X}_u > t\}$. The time-changed process $\widetilde{J}_t = J(\gamma_t)$ is still a Markov process and it is not hard to see that the life time of this Markov chain is distributed as $-I_a$. Hence the distribution of $-I_a$ is of phase–type on $(0, \infty)$ and it follows that φ_a^- is a ratio of two polynomials. Since the negative jumps of X form a compound Poisson process, it is well known (e.g. [12]) that in this case $\mathbb{P}(I_a = 0)$ is non zero iff $X^{(+)}$ is a subordinator. Thus, if $X^{(+)}$ is a subordinator, $\varphi_a^-(\infty) > 0$ and $\#\mathcal{I}^{(-)} = \#\mathcal{J}^{(-)}$.

Writing $I_a^{(+)} = \inf_{t \leq \eta(a)} X_t^{(+)}$ for the infimum of $X^{(+)}$ of (12) up to $\eta(a)$, we find the following inequalities relating I_a and $I_a^{(+)}$

$$\mathbb{P}(-I_a^{(+)} < x) \cdot a/(a + \lambda^{(-)}) \leq \mathbb{P}(-I_a < x) \leq \mathbb{P}(-I_a^{(+)} < x) \qquad x > 0. \quad (18)$$

Indeed, since $X = X^{(+)} - J^{(-)}$ we find that $\{-I_a < x\}$ implies that $\{-I_a^{(+)} < x\}$. The first inequality follows by noting that

$$\mathbb{P}\left(-I_a < x | \text{ first jump } J^{(-)} \text{ after } \eta(a)\right) = \mathbb{P}(-I_a^{(+)} < x).$$

From (18) we see that $\mathbb{P}(-I_a < x)/\mathbb{P}(-I_a^{(+)} < x)$ is bounded above and away from zero as $x \downarrow 0$. A Tauberian theorem combined with the fact that $-I_a^{(+)}$ has an exponential distribution (e.g. [13]) and φ_a^- is a ratio of two polynomials, yields then that $\varphi_a^-(s) \sim \text{const}/s$ as $s \to \infty$ and hence that $\#\mathcal{I}^{(-)} = \#\mathcal{J}^{(-)} + 1$.

2. Recall that $s \mapsto \varphi_a^{\pm}(s)$ are analytic in $\pm\Re(s) < 0$ and don't vanish in $\pm\Re(s) \leq 0$ respectively. Combining with the fact from 1. that φ_a^- is a polynomial with $\varphi_a^-(0) = 1$, we deduce that $\varphi_a^-(s)$ is equal to the right-hand side of (15) on $\Re(s) \geq 0$. See for a different, spectral proof of this statement Section 5.3.

3. Follows from straightforward Laplace inversion of (15) QED

Remark 4 The assumption of distinct roots is only made for convenience; indeed, when the equation $\kappa(s) = a$ has multiple roots, let $n^{(-)}$ denote the number of *different* roots with positive/negative real part and $m^{(-,j)}$ the multiplicity of a root ρ_j with $j \in \mathcal{I}^{(-)}$. Then we find that for $k = 1, \ldots, m^{(-,j)}$

the coefficient $A^-_{j,k}$ of $(-\rho_j)^k/(s-\rho_j)^k$ in the partial fraction decomposition of $\kappa^-_a(s) - \varphi^-_a(\infty)$ is given by

$$A^-_{j,k} = \frac{1}{(m-k)!}\frac{\mathrm{d}^{m-k}}{\mathrm{d}s^{m-k}}\frac{\kappa^-_a(s)(s-\rho_j)^m}{(-\rho_j)^k}\bigg|_{s=\rho_j} \qquad \text{with } m = m^{(-,j)}.$$

By straightforward Laplace inversion, we conclude that

$$\mathbb{P}(-I_a \in \mathrm{d}x) = \sum_{j=1}^{n^{(-)}}\sum_{k=1}^{m^{(-,j)}} A^-_{j,k}(-\rho_j)\frac{(-\rho_j x)^{k-1}}{(k-1)!}\mathrm{e}^{\rho_j x}\mathrm{d}x \qquad x>0.$$

Example For a spectrally positive Lévy process, (15) yields $\kappa^-_a(s) = \frac{\rho_-}{\rho_- - s}$, where ρ_- is the unique negative root of (13). For Kou's jump-diffusion [34] with two-sided exponential jumps, (15) yields $\kappa^-_a(s) = \frac{\rho_1\rho_2}{(\rho_1-s)(\rho_2-s)}\frac{\mu_-+s}{\mu_-}$, where ρ_1, ρ_2 are the negative roots and μ_- is the rate of negative jumps. These explicit expressions are at the root of various explicit computations and approximations in the literature on ruin probabilities and first-time passage barrier options.

Example (**Ruin probabilities**). For Lévy processes X of the form (12), equation (16) yields an explicit expression for the ruin probability

$$\mathbb{P}_x(\exists t \leq \eta(a) : X_t < 0) = \mathbb{P}(-I_a > x) = \sum_{j\in\mathcal{I}^{(-)}} A^-_j \mathrm{e}^{\rho_j x}, \qquad x \geq 0 \tag{19}$$

in case the roots $\rho_i, i \in \mathcal{I}^{(-)}$ are all distinct. For multiple roots with negative real part, similar expressions can be derived using Remark 4. This formula generalizes those of [41] who considered X of the form (12) with negative mixed exponential jumps. Also, Erlang approximations of *finite time* ruin probabilities for one sided phase-type Lévy processes (12) may be obtained as well, generalizing those for the classical ruin model of Asmussen, Avram and Usabel [8]. See also the subsection on the American put below.

Now we consider X of the form (2) and obtain an explicit expression for the resolvent of X killed upon entering a negative half line. Analogous as before, write $\mathcal{I}^{(+)} = \{i : \Re(\rho_i) > 0\}$ and $\mathcal{J}^{(+)} = \{i : \Re(\eta_i) > 0\}$ for the set of roots of $\kappa(\rho) = a$ and $a/(a-\kappa(\eta))$ with positive real part respectively.

Since the analytic continuation of the Laplace transform $\hat{F}$ of a (non-defective) phase–type distribution F is a ratio f_1/f_2 of two polynomials f_1, f_2 with degree(f_1)< degree(f_2), we note from (3) that under the model (2) the function κ is the ratio $\tilde{p}/\tilde{q}$ of two polynomials $\tilde{p}, \tilde{q}$ where degree($\tilde{q}$) – degree($\tilde{p}$) is $2, 1$ or 0 according to whether ($\sigma \neq 0$), ($\sigma = 0, \mu \neq 0$) or ($\mu = \sigma = 0$), respectively.

Corollary 1 *Suppose X is a Lévy process of the form (2).*

(1) On the half-plane $\Re(s) \le 0$, the Wiener-Hopf factor φ_a^+ is given explicitly by

$$\varphi_a^+(s) = \frac{\prod_{i\in\mathcal{I}^+}(-\rho_i)}{\prod_{i\in\mathcal{I}^+}(s-\rho_i)} \cdot \frac{\prod_{j\in\mathcal{J}^{(+)}}(s+\eta_j)}{\prod_{j\in\mathcal{J}^{(+)}}(\eta_j)}$$

(2) Moreover, $\#\mathcal{I}^{(+)} = \#\mathcal{J}^{(+)}$ or $\#\mathcal{J}^{(+)}+1$ according to whether ($\mu=\sigma=0$) or not. If the representation $(m^{(\pm)}, \boldsymbol{T}^{(\pm)}, \boldsymbol{\alpha}^{(\pm)})$ of $F^{(\pm)}$ is minimal, $\#\mathcal{J}^{(\pm)} = m^{(\pm)}$.

(3) Supposing the roots ρ of (4) are different, the resolvent of X killed upon entering $(-\infty, k]$ is for $k<0$ and $y>k$ given by

$$\mathbb{P}(X_{\eta(a)} \in \mathrm{d}y, \eta(a) < T(k))/\mathrm{d}y = \sum_{i\in\mathcal{I}^{(+)}}\sum_{j\in\mathcal{I}^{(-)}} \frac{A_i^+A_j^-(-\rho_j\rho_i)}{\rho_j-\rho_i}\mathrm{e}^{-\rho_i y}[\mathrm{e}^{-(\rho_j-\rho_i)k} - \mathrm{e}^{(\rho_j-\rho_i)(-y)^+}], \quad (20)$$

where $\eta(a)$ denotes,as before, an independent exponential random variable with parameter a, $c^+ = \max\{c,0\}$ and $\boldsymbol{A}^+ = (A_i^+, i\in\mathcal{I}^{(+)})$ are the partial fraction coefficients of the expansion of $\varphi_a^+(s) - \varphi_a^+(-\infty)$ into $\rho_i/(\rho_i - s)$ for $i\in\mathcal{I}^{(+)}$ (where φ_a^+ is interpreted in the sense of its analytical continuation).

Proof The first two statements follow as corollary from Lemma 1. For the third statement we note that for $k<0$ the set $\{\eta(a) < T(k)\}$ is the same as $\{I_a > k\}$ and that (from time-reversal) M_a has the same distribution as $X_{\eta(a)} - I_a$, we find that

$$\mathbb{P}(X_{\eta(a)} \in \mathrm{d}y, \eta(a) < T(k)) = \int_0^{-k} \mathbb{P}(-I_a \in \mathrm{d}z)\mathbb{P}(M_a \in \mathrm{d}(z+y)).$$

Inserting the expressions from (16), we find the stated expression. QED

3.2 First passage time for X

The first passage time problem consists in computing the joint moment generating function

$$u_k(x) = u_k(x,a,b) = \mathbb{E}_x[\mathrm{e}^{-aT+b(X_T-k)}] \quad (21)$$

of the crossing time

$$T = T(k) = \inf\{t>0 : X_t \le k\}$$

and of the shortfall $X_T - k$, with k, $a>0$ and b such that $u_k(x)$ is finite. The subscript x in $\mathbb{E}_x$ refers to $X_0 = x$.

At the crossing time $T(k)$, we must either have a downwards jump of X, or the component $\mu t + \sigma W_t$ must take the process X down to the barrier k. Denote by G_0 the event that the last alternative occurs, by G_i, $i = 1, \ldots, m^{(-)}$, the event that the first occurs and the upcrossing of k occurs in phase i, i.e. that $J(X_{T(k)-} - k) = i$ where J is the underlying phase process for the jump causing the upcrossing, and by $M^{(-)}$ the set of all phases during which downcrossing of a level may occur. Thus, calling the state where the Lévy process is moving continuously phase 0, $M^{(-)} = \{1, \ldots, m^{(-)}\}$ if the Brownian component is zero and if the drift points opposite to the barrier; otherwise, $M^{(-)} = \{0, \ldots, m^{(-)}\}$. Let $\pi_i = \mathbb{E}_x[\exp(-aT(k))\mathbf{1}_{G_i}]$ denote the discounted probability of upcrossing in phase i, where $X_0 = x$. Moreover, let $\mathbf{1}_i$ denote a vector of zeros with a 1 on the ith position, $\boldsymbol{\pi} = (\pi_i, i \in M^{(-)})$, and let $\hat{\boldsymbol{f}}^{(-)}[b]$ denote the vector (depending on the phase at the level crossing) of Laplace transforms at b of the absolute shortfall $|X_{T(k)} - k|$. This vector can be analytically continuated to the complex plane except a finite number of poles (the eigenvalues of $\boldsymbol{T}^{(-)}$). This analytic extension will also be denoted by $\hat{\boldsymbol{f}}^{(-)}$. Note that, if $0 \in M^{(-)}$, then the first component of $\hat{\boldsymbol{f}}^{(-)}[b]$ is 1, and the other components are given by $(b\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$ by the phase assumption and if $0 \notin M^{(-)}$, the first component is missing. The next result gives an explicit expression for the moment-generating function $u_k(x)$ in terms of the roots with negative real part.

Proposition 2 *Subject to (12) we have:*

(1) For any nonnegative function f and $x > k$:

$$\mathbb{E}_x[\mathrm{e}^{-aT(k)} f(X_{T(k)} - k)] = \boldsymbol{\pi}\mathcal{G}f \tag{22}$$

where $\mathcal{G}f = (\int_0^\infty f(-z)F_i^{(-)}(\mathrm{d}z), i \in M^{(-)})$ with $F_0^{(-)}(\mathrm{d}z) = \delta_0(\mathrm{d}z)$ and $1 - F_i^{(-)}(z) = \mathbf{1}_i \exp(\boldsymbol{T}^{(-)}z)\mathbf{1}$ for $i \neq 0$.

(2) For $x > k$ the vector $\boldsymbol{\pi}$ satisfies the system

$$\boldsymbol{\pi}\hat{\boldsymbol{f}}^{(-)}[\rho_i] = \mathrm{e}^{\rho_i(x-k)}, \qquad \forall i \in \mathcal{I}^{(-)}. \tag{23}$$

Moreover, assuming all the roots of the equation (4) with negative real part to be distinct the following hold true:

(3) If the representation $(m^{(-)}, \boldsymbol{T}^{(-)}, \boldsymbol{\alpha}^{(-)})$ of $F^{(-)}$ is minimal, $\boldsymbol{\pi}$ is the unique solution of (23).

(4) In particular, $u_k(x)$ defined in (21) is for $x > k$ given by

$$\mathrm{e}^{bx}u_k(x) = u_{k-x}(0) = \varphi_a^-(b)^{-1} \sum_{j \in \mathcal{I}^{(-)}} A_j^- \rho_j \mathrm{e}^{\rho_j(x-k)}/(\rho_j - b). \tag{24}$$

where A_j^- is defined in (17).

Remark 5 Taking Laplace transform of (24) in $x-k$, we recover a formula of [13] $\hat{u}_0(s) = (b-s)^{-1}\left(1 - \frac{\varphi_a^-(s)}{\varphi_a^-(b)}\right)$ for $\Re(s) \geq 0$.

Remark 6 In case the equation (13) has multiple roots with negative real part, expressions similar to (23) – (24) can be derived by approximation and using Remark 4.

Proof of Proposition 2 1–3. Splitting the probability space in $G_0, \ldots, G_{m^{(-)}}$ and using the fact that, conditionally on the phase in which the upcrossing occurs, the time of overshoot $T(k)$ and the shortfall $X_{T(k)} - k$ are independent, yields the decomposition

$$\mathbb{E}_x[\mathrm{e}^{-aT} f(X_T - k)] = \mathbb{E}_x[\mathrm{e}^{-aT}\mathbf{1}_{G_0}] + \sum_{i=1}^{m^{(-)}} \mathbb{E}_x[\mathrm{e}^{-aT}\mathbf{1}_{G_i}]\mathbb{E}_i[f(X_T - k)],$$

where we wrote $T = T(k)$ and respectively used $\mathbb{E}_x, \mathbb{E}_i$ to denote the expectation under $\mathbb{P}$ conditioned on $\{X_0 = x\}$ and G_i. This yields (22). The system (23) is derived by an optional stopping approach. By applying Itô s formula to the function $f(t, X_t) = \exp(-at + bX_t)$ for any a and b with $\Re(b) = 0$ (which ensures that $\kappa(b)$ is well defined), we find that

$$\begin{aligned} M_t &= f(t, X_t) - f(0, X_0) - \int_0^t Gf(s, X_s)\mathrm{d}s \qquad (25) \\ &= \exp(-at + bX_t) - \exp(bX_0) - (\kappa(b) - a)\int_0^t \exp(-as + bX_s)\mathrm{d}s, \end{aligned}$$

is a zero-mean martingale, where $G = \frac{\partial}{\partial t} + \Gamma$ with Γ the infinitesimal generator of $\{X_t, t \geq 0\}$ (note that $Gf(t, X_t) = (\kappa(b) - a)f(t, X_t))$. Applying for $a \geq 0$ Doob's optional stopping theorem with the stopping time $T(k) \wedge t$ and noting that $\sup_t |M_{T(k)\wedge t}|$ is bounded we find $\mathbb{E}_x[M_{T(k)}] = 0$. By a computation as above we can expand this for $x > k$ as

$$0 = \mathrm{e}^{bk}\boldsymbol{\pi}\hat{\boldsymbol{f}}^{(-)}[b] - \mathrm{e}^{bx} - (\kappa(b) - a)\mathbb{E}_x\left[\int_0^{T(k)} \exp(-as + bX_s)\mathrm{d}s\right]. \qquad (26)$$

By analytic continuation, the identity (26) can be extended to the half plane $\Re(b) < 0$ except finitely many poles (the eigenvalues of $\boldsymbol{T}^{(-)}$, since $\boldsymbol{T}^{(-)}$ has eigenvalues with negative real part). By choosing b with $\Re(b) < 0$ to be a root of the equation $\kappa(b) = a$, we find (23). By Lemma 1 the number of equations is equal to the number of unknowns, if the representation of $F^{(-)}$ is minimal. The distinct roots assumption implies then the linear independence of $\hat{\boldsymbol{f}}^{(-)}[\rho_i]$, as proved in Section 5. Hence the "Wald system" (23) is nonsingular, yielding $\boldsymbol{\pi}$.

4. Suppose first $b, a > 0$, and note that $\mathrm{e}^{bx}u_k(x) = u_{k-x}(0)$. Define $A = \{T(k-x) < \eta(a)\}$. The strong Markov property of X applied at $T(k-x)$

together with the memoryless property of the exponential distribution imply that

$$\mathbb{E}[\exp(bI_a)\mathbf{1}_A] = \mathbb{E}[\exp(bX_{T(k-x)})\mathbf{1}_A]\mathbb{E}[\exp\{bI_a\}]$$
$$= \mathbb{E}[\exp(-aT(k-x) + bX(T(k-x)))]\kappa_a^-(b),$$

where $\mathbf{1}_A$ denotes the indicator of the event A. Noting that $A = \{I_a < k - x\}$ and using (16) one finds the formula as stated. By analytic extension, the identity holds for all b for which the right-hand side of (24) is well defined. QED

3.2.1 American put and Erlang approximations

Under the "one-sided phase–type" model (12) and assuming that the roots $\rho_j \in \mathcal{I}^{(-)}$ are different, the value of the American put option for $e^x > e^{k^*} = K\varphi_\delta^-(1)$ can be checked to be given by

$$U^*(x) = K\mathbb{E}_x[e^{-\delta T(k^*)}] - e^{k^*}\mathbb{E}_x[e^{-\delta T(k^*)+X_{T(k^*)}-k^*}]$$
$$= K\sum\nolimits_{j\in\mathcal{I}^{(-)}} e^{\rho_j(x-k^*)}A_j^-/(1-\rho_j), \quad (27)$$

where $\rho_j = \rho_j(\delta)$ denote the roots of $\kappa(\rho) = \delta$ for $\delta = r + a$ (just insert the expressions for k^* and the joint moment-generating function $u_k(x)$ of $T(k^*)$ and $X_{T(k^*)} - k^*$). The important application here is with the parameter $\delta = r + (T-t)^{-1}$, where t, T denote the current and expiration time of a finite expiration option. Recalling that $\kappa(1) = r$ we see that the optimal exercise level $k^* = k^*(t,T)$ is given by

$$\exp(k^*) = K\frac{\delta}{\delta - \kappa(1)}\,\frac{1}{\varphi_\delta^+(1)} = K(r(T-t)+1)\,\frac{1}{\varphi_\delta^+(1)}.$$

As noticed in [17,9], k^* yields a time dependent approximation for the optimal excercise boundary of an American put with expiration time T, which may be checked to be asymptotically exact when $t \to -\infty$ and also when $t \to T$.

We can also obtain the value of an American put on a stock paying proportional dividends. Indeed, the value of an American put option with payoff (5) on a stock paying dividends at rate $q \geq 0$ can be found by choosing $\mathbb{P}$ such that $\kappa(1) = r - q$ (instead of r) and by replacing everywhere in (27) the parameters (r, a) by $(r - q, a + q)$.

Further refinements under the "two sided phase–type" model (2) may be obtained by Erlangizing the expiration time, a method which goes back at least as far as S. Ross [45], and which was first implemented in mathematical finance by Carr [16] (see also [9,36]). By this approach, one can obtain a sequence of analytic formulae that converges pointwise to the price of the American put with finite time of expiration T, extending thus the spectrally negative results in [9,10]. We give an outline how to obtain the first approximation, known as the "Canadized" American put option. Letting $\eta(T^{-1})$ denote an independent exponential random variable with mean T, by standard optimal stopping theory (see the argument of Theorem 1), one shows that the optimal stopping time for this option is again of the form $T(k)$ for some $k < \log K$. Computing the value function U_1^* thus boils down to evaluating

$$\mathbb{E}_x[\mathrm{e}^{-r(T(k)\wedge\eta(T^{-1}))}(K - \mathrm{e}^{X_{T(k)\wedge\eta(T^{-1})}})^+] =$$
$$\mathbb{E}_x[\mathrm{e}^{-qT(k)}(K - \mathrm{e}^{X_{T(k)}})^+] + \frac{1}{qT}\int_k^{\log K}(K - \mathrm{e}^z)\mathbb{P}_x(X_{\eta(q)} \in \mathrm{d}z, \eta(q) < T(k)),$$

where $q = r + T^{-1}$, followed by a one-dimenional optimization (or continuous/smooth fit) to find the optimal level k_1^*. The evaluation of the second term in the display uses the resolvent of X killed upon entering $(-\infty, k]$, from Proposition 1. If $\sigma \neq 0$ and the roots of (4) are distinct, the result reads as

$$U_1^*(x) = \begin{cases} \frac{K}{qT} - \mathrm{e}^x + c(x) + \frac{K}{qT}\sum_{i,j}\frac{A_j^- A_i^+(-\rho_j\rho_i)}{\rho_j - \rho_i}[d_i(x) - e_{ij}(x)] & x \in (k, \log K) \\ c(x) + \frac{K}{qT}\sum_{i,j}\frac{A_j^- A_i^+(-\rho_j\rho_i)}{\rho_j - \rho_i}[d_j(x) - e_{ij}(x)] & x \geq \log K, \end{cases}$$

where the sum is over $i \in \mathcal{I}^{(+)}$ and $j \in \mathcal{I}^{(-)}$, $c(x) = K\frac{r}{q}\sum_{j\in\mathcal{I}^{(-)}} A_j^- \mathrm{e}^{\rho_j(x-k)}$, $d_i(x) = \frac{\mathrm{e}^{\rho_i x}K^{-\rho_i}}{\rho_i(1-\rho_i)}$, $e_{ij}(x) = d_i(k)\mathrm{e}^{\rho_j(x-k)}$ and the optimal exercise level $k = k_1^*$ is

$$\exp k_1^* = K\sup\left\{x \leq \log K : rT = \sum_{i\in\mathcal{I}^{(+)}} A_i^+ x^{\rho_i}/(\rho_i - 1)\right\}. \tag{28}$$

By the definition of the A_i^+ as partial fraction coefficients it follows that $\sum\frac{A_i}{\rho_i - 1} = \varphi^+_{r+T^{-1}}(1) - 1$, which is larger than rT, and we note that $\exp k_1^* < K$.

3.3 First passage time for Y

We now consider the first passage time problem for Y, which, analogously, consists in computing the joint moment generating function

$$v_k(y) = v_k(y, a, b) = \mathbb{E}_y[\mathrm{e}^{-a\tau + b(Y_\tau - k)}] \tag{29}$$

of the crossing time

$$\tau = \tau(k) = \inf\{t > 0 : Y_t \geq k\}$$

and of the overshoot $Y_\tau - k$, with k, $a \geq 0$, and where b is such that $v_k(y)$ is finite. Under the measure $\mathbb{P}_y$, the process Y starts in y.

Analogously to the previous section 3.2, we note that at the crossing time $\tau(k)$, we must either have a downward jump of X, or the component $\mu t+\sigma W_t$ must take the process Y to the barrier k. Denote by $M^{(-)}$ the set of all phases during which upcrossing may occur (again calling the non-jumping time phase 0). Let $\widetilde{\pi}_i = \mathbb{E}_y[\mathrm{e}^{-a\tau}\mathbf{1}_{H_i}]$ denote the (discounted) probability of upcrossing in phase i, i.e. that $J(k-Y_{\tau(k)-})=i$, where J is the underlying phase process for the jump causing the upcrossing. Analogously as before we write $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi}_i, i\in M^{(-)})$, and let $\hat{\boldsymbol{f}}^{(-)}[b]$ denote the vector (depending on the initial starting state) of analytic continuations of Laplace transforms at b of the overshoot $Y_{\tau(k)}-k$. Let $L_t=\sup_{0\le s\le t}X_s\vee y$ be the running supremum of X, with L^c_t the continuous part of L and $\Delta L_t = L_t - L_{t-}$ the jump of L at time t. Introduce the dummy-variables $\delta_0=\mathbb{E}_y[\int_0^{\tau(k)}\exp(-as)\mathrm{d}L^c_s]$ and

$$\delta_i = \mathbb{E}_y\left[\sum_{0<s\le\tau(k)}\exp(-as)\mathbf{1}_{\{\Delta L_s>0,H_j\}}\right],\ j=1,\ldots,m^{(+)},$$

where H_j is the event of crossing the supremum in phase j. Denote by $\boldsymbol{\delta}$ the row vector $\boldsymbol{\delta}=(\delta_i, i\in M^{(+)})$ and write $\boldsymbol{g}[\rho]=(g[\rho]_i, i\in M^{(+)})$ with $g[\rho]_0=\rho$ and $g[\rho]_i=\rho\mathbf{1}_i(-\rho\boldsymbol{I}-\boldsymbol{T}^{(+)})^{-1}\mathbf{1}$ and let $p=\#\mathcal{I}^{(+)}+\#\mathcal{I}^{(-)}$ denote the number of roots of $\kappa(\rho)=a$.

Proposition 3 *Subject to (2), the joint moment generating function $v_k(y)$ defined in (29) is for $y\in[0,k)$ given by*

$$v_k(y)=\widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-b].$$

where $\widetilde{\boldsymbol{\pi}}=(\widetilde{\pi}_i, i\in M^{(-)})$ and $\boldsymbol{\delta}=(\delta_i, i\in M^{(+)})$ solve the system

$$\mathrm{e}^{-\rho_i y}=\mathrm{e}^{-\rho_i k}\widetilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[\rho_i]-\boldsymbol{\delta}\boldsymbol{g}[\rho_i]\qquad i=1,\ldots,p. \tag{30}$$

If all roots ρ_i of $\kappa(\rho)=a$ are distinct and the vectors $\widetilde{\boldsymbol{k}}^{(i)}:=(\mathrm{e}^{-\rho_i k}\hat{\boldsymbol{f}}^{(-)}[\rho_i]',\boldsymbol{g}[\rho_i]')$, $i=1\ldots,p$, are linearly independent, $(\widetilde{\boldsymbol{\pi}},\boldsymbol{\delta})$ uniquely solve the system (30).

Proof The proof of the first part is analogous to the proof of the second part of Proposition 2 and left to the reader. To compute the vector $\widetilde{\boldsymbol{\pi}}$, we apply the optional stopping approach to the reflected process Y, using the martingale introduced by Kella and Whitt [32]. Note that L^c and $\Delta L_t = L_t - L_{t-}$ have finite expected variation resp. finite number of jumps in each finite time interval. From Kella and Whitt [32] we find then that for $a>0,\gamma\in\mathrm{i}\mathbf{R}$

$$\begin{aligned}N_t &= (\kappa(-\gamma)-a)\int_0^t(-as+\gamma Y_s)\mathrm{d}s+\exp(\gamma Y_0)-\exp(-at+\gamma Y_t)\\ &\quad+\gamma\int_0^t\exp(-as)\mathrm{d}L^c_s+\sum_{0<s\le t}\exp(-as)[1-\exp(-\gamma\Delta L_s)]\end{aligned}$$

is a zero mean martingale (where we used that if ΔL_s or $\mathrm{d}L_s$ is positive then $Y_s = 0$). Applying, as before, Doob's optional stopping theorem with the stopping time $\tau(k) \wedge t$ and straightforwardly checking that $|N_{\tau(k)\wedge t}|$ can be dominated by an integrable function, we find $\mathbb{E}_y[N_{\tau(k)}] = 0$. Then, expanding $\mathbb{E}_y[N_{\tau(k)}] = 0$ for $y < k$ leads to

$$0 = (\kappa(-\gamma) - a)\mathbb{E}_y\left[\int_0^{\tau(k)} \exp(-as + \gamma Y_s)\mathrm{d}s\right] + \mathrm{e}^{\gamma y} - \mathrm{e}^{\gamma k}\tilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-\gamma]$$
$$+ \gamma\delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i(1 - \hat{\boldsymbol{f}}^{(+)}[\gamma]_i). \tag{31}$$

By analytic continuation, the identity (31) can be extended to hold for γ in the complex plane except finitely many poles (the eigenvalues of $-\boldsymbol{T}^{(-)}, \boldsymbol{T}^{(+)}$) Letting ρ_j to be a root of $\kappa(\rho) = a$, we find the system (30). Since for minimal representations of $F^{(\pm)}$ we have that $M^{(\pm)} = \mathcal{I}^{(\pm)}$, the number of unknowns is equal to the number of equations, the last assertion follows. QED

Denote by $\widetilde{\boldsymbol{S}} = \begin{pmatrix}\widetilde{\boldsymbol{S}}_1\\ \widetilde{\boldsymbol{S}}_2\end{pmatrix}$ is the $p \times p$ matrix whose first $m^{(-)} + 1$ rows $\widetilde{\boldsymbol{S}}_1$ are columnwise given by $\mathrm{e}^{-\rho_j k}\hat{\boldsymbol{f}}^{(-)}[\rho_j]$ and whose last $m^{(+)} + 1$ rows $\widetilde{\boldsymbol{S}}_2$ are columnwise given by $\boldsymbol{g}[\rho_j]$. If $\sigma \neq 0$ and $\widetilde{\boldsymbol{S}}$ has full rank, the solution of the system (30) is in matrix notation form:

$$(\tilde{\boldsymbol{\pi}} \ - \boldsymbol{\delta}) = (\mathrm{e}^{-\rho_1 y} \ \ldots \ \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}, \tag{32}$$

and from Proposition 3 we conclude then that,

$$v_k(y) = (\mathrm{e}^{-\rho_1 y} \ \ldots \ \mathrm{e}^{-\rho_p y})\widetilde{\boldsymbol{S}}^{-1}\hat{\boldsymbol{f}}_o^{(-)}[-b], \qquad y \in [0, k),$$

where $\hat{\boldsymbol{f}}_o^{(-)}[-b]$ denotes the column vector of Laplace transforms of the overshoots over k prolonged by 0's. Therefore, $v_k(y) = \sum_{i=1}^p \mathrm{e}^{-\rho_i y}A_i$ is a linear combination of the exponentials, with the vector $\boldsymbol{A}$ satisfying the linear system

$$\widetilde{\boldsymbol{S}}\boldsymbol{A} = \hat{\boldsymbol{f}}_o^{(-)}[-b]. \tag{33}$$

Replacing in above paragraph the vectors $\tilde{\boldsymbol{k}}_1^{(j)}$ by $(\rho_j\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$ if $\sigma = 0$ and $\mu \geq 0$ and $\tilde{\boldsymbol{k}}_2^{(j)}$ by $(-\rho_j\boldsymbol{I} - \boldsymbol{T}^{(+)})^{-1}\mathbf{1}$ if $\sigma = 0$ and $\mu \leq 0$, the result (33) for the corresponding matrix $\widetilde{\boldsymbol{S}}$ remains valid.

To connect to other results in the literature, we reformulate now the system (33) for $\boldsymbol{A}$ in terms of the eigenvalues of the matrices $\boldsymbol{T}^{(\pm)}$, allowing at the same time for a general Jordan structure. Let $\eta_j^{(\pm)}$, where $j = 1, \ldots, n^{(\pm)}$, denote the roots of $a/(a - \kappa(\pm\eta)) = 0$ with negative real part and denote their respective multiplicities by $m^{(\pm,j)}$. Note that if the representation $(m^{(\pm)}, \boldsymbol{T}^{(\pm)}, \boldsymbol{\alpha}^{(\pm)})$ of $F^{(\pm)}$ is minimal, the $\eta_j^{(\pm)}$ are precisely the eigenvalues of $\boldsymbol{T}^{(\pm)}$ respectively.

Consider the system of p equations for $A_1, \ldots, A_p$:

$$\sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i k} \frac{1}{(\rho_i - \eta_j^{(-)})^{\ell}} = \frac{1}{(-b - \eta_j^{(-)})^{\ell}} \tag{34}$$

$$\sum_{i=1}^{p} A_i \rho_i \frac{1}{(-\rho_i - \eta_j^{(+)})^{\ell}} = 0, \tag{35}$$

where in (34) and (35) $\ell = 1, \ldots, m^{(\mp,j)}$, $j = 1, \ldots, n^{(\mp)}$ respectively and in addition $\ell = 0$ in (34) if $\sigma \neq 0$ or $\mu < 0$ [$l = 0$ in (35) if $\sigma \neq 0$ or $\mu > 0$].

Proposition 4 *Under (2), assuming the roots ρ_i of $\kappa(\rho) = a$ to be distinct and the vectors $\widetilde{\boldsymbol{k}}^{(j)}$ to be linearly independent, the system (34) - (35) has a unique solution $A_1, \ldots, A_p$ and we have*

$$v_k(y) = \sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i y}, \qquad y \in [0, k). \tag{36}$$

In Section 5.3 we provide an independent proof by solving the corresponding integro-differential equation.

3.3.1 The Russian option

Now we turn to the explicit solution of the optimal stopping problem connected to the pricing of the Russian option. Recall $\hat{\boldsymbol{f}}_o^{(-)}[-b]$ denotes the column vector of Laplace transforms of the overshoots of Y over k prolonged by 0's. Combining Theorem 1 with the results of the foregoing section leads to the following statement:

Corollary 2 *Let X be of the form (2) and satisfy (EMM). Assume that the roots of (13) are distinct and the vectors $\widetilde{\boldsymbol{k}}^{(j)}$,$j = 1, \ldots, p$, are linearly independent. Then the price of the Russian option is given by*

$$v^*(y) = \begin{cases} \mathrm{e}^{k^*} \widetilde{\boldsymbol{\pi}}(y, k^*) \hat{\boldsymbol{f}}^{(-)}[-1] = \mathrm{e}^{k^*} \sum_{i=1}^{p} A_i(k^*) \mathrm{e}^{-\rho_i y}, & y \in [0, k^*); \\ \mathrm{e}^{y}, & y \geq k^*, \end{cases}$$

where ρ_i are the roots of $a = \kappa_1(\rho) = \kappa(\rho + 1) - \kappa(1)$ and the $A_i = A_i(k^*)$ are given in (36) and are, just as $\widetilde{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{f}}^{(-)}$, computed under the measure $\mathbb{P}^{(1)}$. The optimal level k^* satisfies the following:

(1) If $\mu \geq \sigma = 0$ and $\iota := \frac{\lambda^{(-)}}{a + \lambda^{(-)}} \boldsymbol{\alpha}^{(-)} \hat{\boldsymbol{f}}^{(-)}[-1] > 1$, k^* is a positive root of

$$(\mathrm{e}^{-\rho_1 k}, \ldots, \mathrm{e}^{-\rho_p k}) \widetilde{\boldsymbol{S}}^{-1} \hat{\boldsymbol{f}}_o^{(-)}[-1] = \sum_{i=1}^{p} A_i \mathrm{e}^{-\rho_i k} = 1; \tag{37}$$

(2) If $\mu < -a$ and $\sigma = 0$, k^* is a positive root of

$$(\rho_1 \mathrm{e}^{-\rho_1 k}, \ldots, \rho_p \mathrm{e}^{-\rho_p k}) \widetilde{\boldsymbol{S}}^{-1} \hat{\boldsymbol{f}}_o^{(-)}[-1] = \sum_{i=1}^{p} \rho_i A_i \mathrm{e}^{-\rho_i k} = -1; \tag{38}$$

(3) If $\sigma \neq 0$, k^* is positive and uniquely determined by (38).

In the literature equations (37) and (38) are called the conditions of *continuous* (e.g. [44]) and *smooth fit* respectively. If for the process Y 0 is *regular* for $(0, \infty)$ (that is $\mathbb{P}(\tau_{(0,\infty)} = 0 | Y_0 = 0) = 1$, where $\tau_{(0,\infty)}$ is the first time Y enters $(0, \infty)$), it satisfies continuous fit for all levels $k > 0$ and to determine the optimal level an extra condition is needed. We observe from above result that, if the optimal level is positive, in this case the optimal level satisfies the condition of smooth fit. If 0 is *irregular* for $(0, \infty)$ for Y and the optimal level is positive then the optimal level satisfies the condition of continuous fit.

Example Consider the case where X is a Brownian motion with drift. Denote by $\rho_1 < 0 < \rho_2$ the two roots of

$$\kappa_1(s) = \kappa(s+1) - r = \frac{\sigma^2}{2} s^2 + \left(r + \frac{\sigma^2}{2} \right) s - r = a.$$

Since $\hat{\boldsymbol{f}}_0^{(-)}[1] = (1, 0)'$ and $\boldsymbol{S} = \begin{pmatrix} \mathrm{e}^{-\rho_1 k} & \mathrm{e}^{-\rho_2 k} \\ -\rho_1 & -\rho_2 \end{pmatrix}$, we find by adding (37) to (38) that the optimal level k^* is given by

$$\exp\left((\rho_2 - \rho_1) k^* \right) = \frac{\rho_1(\rho_2 + 1)}{\rho_2(\rho_1 + 1)},$$

which is the formula found by Shepp and Shiryaev [47].

In the proof of the corollary we will use the following auxiliary result, which is proved in Section 5.

Lemma 2 *Under the assumptions of Corollary 2, the following hold true:*

(1) If $\mu \geq \sigma = 0$, then $v_k(0) \to \iota$ and if $\mu < \sigma = 0$, then $v_k'(y)|_{y=k^-} \to -\frac{a}{\mu}$ as $k \downarrow 0$.
(2) If $\sigma \neq 0$, then the function $k \mapsto v_k'(y)|_{y=k^-}$ is continuous and increasing on $(0, \infty)$ with $\lim v_k'(y)|_{y=k^-} = 0$ or > 1 if $k \downarrow 0$ and $k \to \infty$ respectively.

Proof of Corollary 2 The only statements left to prove are the ones on the optimal level k^*, the rest follows from Theorem 1 and Propositions 3 and 4. Let us first consider the situation that $\mu \leq 0 = \sigma$. In this case we have that ι is equal to $\lim_{k \downarrow 0} v_k(0)$. Hence, if $\iota > 1$, then $v^*(0) > 1$ and $k^* > 0$. Since $y \mapsto v^*(y) = \mathrm{e}^{k^*} v_{k^*}(y)$ is convex and hence continuous on $(0, \infty)$, it follows that the optimal level k^* satisfies $v_{k^*}(k^*-) = 1$, which is (37).

Now let us consider the case $\mu < -a$ or $\sigma \neq 0$. Note first that if k^* is an optimal level, $k \mapsto \mathrm{e}^k v_k(y)$ is maximized in $k = k^*$ for all y. Thus $\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(0)\right)|_{k=0^+} \leq 0$ is a necessary condition for $k^* = 0$ to be optimal. Moreover, if $k^* > 0$, then it satisfies

$$\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(y)\right)\Big|_{k=k^*} = 0 \qquad \text{for all } y < k^*$$

and in particular $\frac{\partial}{\partial k}\left(\mathrm{e}^k v_k(k^*-)\right)|_{k=k^*} = 0$. Secondly, we note that in this case Y is regular for $(0,\infty)$, that is the first time Y enters $(0,\infty)$ is almost surely 0, which yields the identity

$$\mathrm{e}^k v_k(k^-) = \mathrm{e}^k v_k(k) = \mathrm{e}^k \qquad \text{for all } k > 0. \tag{39}$$

Differentiating (39) with respect to k we find

$$\frac{\partial}{\partial z}\left(\mathrm{e}^z v_z(k^-)\right)\Big|_{z=k} + \mathrm{e}^k \frac{\partial}{\partial y} v_k(y)\Big|_{y=k^-} = \mathrm{e}^k. \tag{40}$$

Since, by Lemma 2, $\lim_{k\downarrow 0} v_k'(y)|_{y=k^-} < 1$, we deduce that in this case $k^* > 0$ and that k^* is a positive root of $v_k'(y)|_{y=k^-} = 1$, which is the equation (38). If $\sigma \neq 0$, we see from Lemma 2(ii) that there is a unique $c > 0$ such that $v_c'(c^-) = 1$. QED

The "Canadized" Russian option is understood to be the Russian option with an independent exponential random variable $\eta(\lambda)$ as expiration, an analog of the Canadized American put. It can be considered as a first approximation to the Russian option with finite expiration $1/\lambda$. See [36,10]. The value of the Canadized Russian option is given by $V_c^*(x,m) = \mathrm{e}^x v_c^*(m-x)$, where v_c^* is the value function of the optimal stopping problem

$$v_c^*(y) = \sup \mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau\wedge\eta(\lambda))+Y_{\tau\wedge\eta(\lambda)}}].$$

where the supremum runs over τ in $\mathcal{T}$. Mimicking the proof of Theorem 1, we check that again the optimal stopping time is of the form (9). The quantities $\tilde{\boldsymbol{\pi}}$ and $\boldsymbol{\delta}$ are now understood to be taken under the measure $\mathbb{P}_y^{(1)}$. Note that $\kappa_1(-1) = \kappa(0) - \kappa(1) = -r$ (see Appendix A). Then we can read off from equation (31) that for $y < k$ and with $\gamma = \lambda/(a+\lambda+r)$, we have that

$$\mathbb{E}_y^{(1)}[\mathrm{e}^{-a(\tau_k\wedge\eta(\lambda))+Y_{\tau_k\wedge\eta(\lambda)}}] = \tilde{\boldsymbol{\pi}}\hat{\boldsymbol{f}}^{(-)}[-1](1-\gamma)\times\mathrm{e}^k + \gamma(\mathrm{e}^y + \delta_0 + \sum_{i=1}^{m^{(+)}} \delta_i(1-\hat{\boldsymbol{f}}^{(+)}[1]_i)).$$

By optimization of this expression over all levels $k \geq 0$ (or by smooth/continuous fit), we find $v_c^*(y)$.

Example Let X be given by a jump-diffusion where the jumps have a negative hyper-exponential distribution. In the general setting we choose $\sigma >$

0, $\lambda^{(+)} = 0$, $-\boldsymbol{T}^{(-)} = \mathrm{diag}(\beta_1, \ldots, \beta_n)$, β_i different, and $\boldsymbol{\alpha}^{(-)} = (\alpha_1, \ldots, \alpha_n)$. From Appendix A, we find that the parameters of X under $\mathbb{P}^{(1)}$ are given by

$$\widetilde{\mu} = \mu + \sigma^2, \qquad \widetilde{\lambda}^{(+)} = 0, \qquad -\widetilde{\boldsymbol{T}}^{(-)} = \mathrm{diag}(1+\beta_1, \ldots, 1+\beta_n)$$

$$\widetilde{\lambda}^{(-)} = \lambda^{(-)}\boldsymbol{\alpha}^{(-)}(\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)} = \lambda^{(-)}\sum_{i=1}^{n}\alpha_i\frac{\beta_i}{\beta_i+1}$$

$$\widetilde{\boldsymbol{\alpha}}^{(-)} = \boldsymbol{\alpha}^{(-)}\mathrm{diag}(k_1, \ldots, k_n)/\hat{F}^{(-)}[1] = \frac{1}{\sum_{i=1}^{n}\frac{\alpha_i\beta_i}{\beta_i+1}}(\frac{\alpha_1\beta_1}{\beta_1+1}, \ldots, \frac{\alpha_n\beta_n}{\beta_n+1}),$$

where $\boldsymbol{k} = (\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\boldsymbol{t}^{(-)}$. Let ρ_i be the roots of $\kappa_1(s) = a$ and note that they are all distinct. Then the price $V^*(x,m)$ of the Russian option is given by $V^*(x,m) = \mathrm{e}^x v^*(m-x)$ where $v^*(y) = \mathrm{e}^y$ for $y \geq k^*$ and

$$v^*(y) = \mathrm{e}^{k^*}\textstyle\sum_{i=0}^{n+1} A_i\mathrm{e}^{-\rho_i y} \qquad 0 \leq y < k^* \tag{41}$$

with the A_i and k^* are determined by

$$\sum_{i=0}^{n+1} A_i\mathrm{e}^{-\rho_i k^*} = 1 \qquad \sum_{i=0}^{n+1} A_i\rho_i = 0 \qquad \sum_{i=0}^{n+1} A_i\rho_i\mathrm{e}^{-\rho_i k^*} = -1$$
$$\sum_{i=0}^{n+1} A_i\rho_i\mathrm{e}^{-\rho_i k^*}\frac{1}{1+\beta_j+\rho_i} = \frac{1}{1+\beta_j-1} \qquad (j = 1, \ldots, n).$$

The first equation in the second line is smooth fit condition which determines k^*. Write now $C_i = A_i\mathrm{e}^{-\rho_i k}$ then we can rewrite the previous system as

$$1 = \sum_{i=0}^{n+1} C_i = -\sum_{i=0}^{n+1} C_i\rho_i = \sum_{i=0}^{n+1} C_i\frac{\beta_j}{1+\beta_j+\rho_i} \quad (j = 1, \ldots, n)$$

to find the B_i and then to find the k^* the equation $\sum_{i=0}^{n+1} C_i\rho_i\mathrm{e}^{\rho_i k^*} = 0$. By a partial fraction argument based on the rational function

$$\frac{\prod_{j=0}^{n+1}(\rho_j+1)}{\prod_{j=1}^{n}(-\beta_j)}\frac{\prod_{j=1}^{n}(s-\beta_j)}{\prod_{j=0}^{n+1}(s+\rho_j+1)},$$

we see that

$$A_i\mathrm{e}^{-\rho_i k} = C_i = \frac{\prod_{j=0}^{n+1}(\rho_j+1)}{\prod_{j=0,j\neq i}^{n+1}(\rho_j-\rho_i)}\frac{\prod_{j=1}^{n}(1+\rho_i+\beta_j)}{\prod_{j=1}^{n}\beta_j}.$$

The found formula for the value of the Russian option conincides with the results from [42].

4 Regime-switching Lévy processes

4.1 Introduction

In this section, we study a certain class of regime-switching Lévy processes following an approach based on embedding first the Lévy model into a *continuous regime switching Brownian motion*, as proposed in [3] (see also [5], [6]).

Definition. A regime switching phase–type Lévy process X is a semi-markov process to which is associated an ergodic finite state space Markov process J such that, conditional on $J_t = j$, X_t is a Lévy model of the form (2) with parameters depending on j. In the case of no jumps the process is called a regime switching Brownian motion.

The trick of passing from a phase–type regime switching Lévy process to a regime switching Brownian motion is to level out the positive jumps to sample path segments with slope $+1$ and the negative jumps to sample path segments with slope -1, and add an extra phase, say 0, for the "regular time" when the process evolves continuously. This embeds a process with phase–type jumps X in a continuous Markov additive process (J, X'), or regime switching Brownian motion, where the Markov component J_t is in phase 0 at a regular time and gives the current phase of the jump otherwise.

For a general regime switching Lévy process Z, let us denote by $\boldsymbol{F}_t[s]$ the $p \times p$ matrix with ijth element $\mathbb{E}_i[e^{sZ_t}; J_t = j]$. Then ([6] p. 41) $\boldsymbol{F}_t[s] = e^{t\boldsymbol{K}[s]}$ where

$$\boldsymbol{K}[s] \ = \boldsymbol{Q} + \{\kappa^{(j)}(s)\}_{\text{diag}} \tag{42}$$

and $\kappa^{(j)}(s)$ is the Lévy exponent in phase j. Many of the computations involving regime switching Lévy processes reduce to finding the eigenstructure of the matrix $\boldsymbol{K}[s]$. For example, Asmussen & Kella [7] solved the first passage time problem for reflected regime switching Brownian motion by introducing the (row) vector martingale

$$\mathrm{e}^{bY_t - at}\mathbf{1}_{J_t} - \mathrm{e}^{by}\mathbf{1}_{J_0} - b\int_0^t \mathrm{e}^{-au}\mathbf{1}_{J_u} dL_u - \int_0^t \mathrm{e}^{bY_u - au}\mathbf{1}_{J_u} du \ \ \boldsymbol{K}[b]$$

where $\mathbf{1}_i$ denotes a (row) vector with a 1 in the ith coordinate and 0's everywhere else and L represents the local time at 0. To use the vector martingale, one forms first scalar martingales obtained by choosing $b = \rho_j$ such that $\boldsymbol{K}[b]$ is singular and by multiplying the vector martingale by the right eigenvectors $\boldsymbol{h}^{(j)}$ of $\boldsymbol{K}[\rho_j]$, with the effect that the last term falls down, yielding the family of scalar martingales

$$M_t^{(j)} \ = \mathrm{e}^{-at+\rho_j Y_t}\boldsymbol{h}_{J_t}^{(j)} - \ \mathrm{e}^{\rho_j y}\boldsymbol{h}_0^{(j)} - b\int_0^t \mathrm{e}^{-as}\boldsymbol{h}_{J_s}^{(j)}\,\mathrm{d}L_s,$$

to which one may apply the optional stopping theorem.

4.2 First passage for regime switching reflected Lévy processes

Let now X be a regime-switching Lévy process with two regimes, where the regimes of X switch from 1 to 2 and vice versa at rates η_1 and η_2 respectively. We denote by $J \in \{1,2\}$ the corresponding Markov-process indicating the current regime of X. If $J_t = i \in \{1,2\}$, $X = X^i$ is of the form (2) with parameters μ_i, σ_i, λ_i, $\boldsymbol{T}_i^{(\pm)}$ and $\boldsymbol{\alpha}_i^{(\pm)}$. We study the first passage problem for $Y = \overline{X} - X$, X reflected at its supremum. In analogogy with the foregoning section, $M^{j(-)}$ will denote all states of the underlying phase processes of the jumps of Y^j causing the upcrossing of levels. Then we are interested in the joint moment generating function

$$v_k^{(i,j)}(y) = v_k^{(i,j)}(y,a,b) = \mathbb{E}_{y,i}[\exp(-a\tau + b(Y_\tau - k)\mathbf{1}_{\{J_\tau = j\}}]$$

of the crossing time

$$\tau = \inf\{t \geq 0 : Y_t \geq k\}$$

and the overshoot $Y_\tau - k$. Here $i, j \in \{1,2\}$, $a \geq 0$ and b such that $v_k^{(i,j)}$ is finite. $\mathbb{E}_{i,y}$ denotes the measure under which $\{Y_0 = y, J_0 = i\}$. By the Markov property, we find as before that the moment-generating function $v_k^{(i,j)}$ is given by

$$v_k^{(i,j)}(y) = \boldsymbol{\pi}^{(i,j)}\hat{\boldsymbol{f}}^{j(-)}[-b],$$

where $\hat{\boldsymbol{f}}^{j(-)}[-b]$ is the Laplace-transform of overshoots $Y_\tau^j - k$, with Y^j denoting Y being in regime $j \in \{1,2\}$, and where

$$\boldsymbol{\pi}^{(i,j)} = (\mathbb{E}_{i,y}[\mathrm{e}^{-a\tau}\mathbf{1}_{G_{j,j'}}], \qquad j' \in M^{j(-)}), \tag{43}$$

with $G_{j,j'} = \{J_\tau = j, \text{level } k \text{ crossed in phase } j'\}$. We embed now the regime-switching Lévy process X into a fluid process X' by leveling out positive jumps of X to sample path segments of X' with slope $+1$, and negative jumps of X to sample path segments of X' with slope -1. More precisely, the phase process $J' = (J, \widetilde{J})$ is defined as follows. The first component $J(t) = i \in \{1,2\}$, indicates that the regime-switching Lévy process X is at time t in regime i. The second component $\widetilde{J}$ takes value $\widetilde{J}(t) = j \in \{1,\ldots,m_i^{(+)}\}$ if, at time t, X' is in one of the segments with slope $+1$ (such that the state of the uncerlying phase process corresponding to the upward jump of X^i is j), and value $j \in \{-1,\ldots,-m_i^{(-)}\}$ if, at time t, X' is in one of the segments with slope -1 (such that the phase of the corresponding downward jump of X^i is

j); when at time t the X'–process operates according to the Lévy exponent $s\mu_i + s^2\sigma_i^2/2$, we let $\widetilde{J}(t) = 0$. The resulting process is a particular case of a regime switching Brownian motion.

Let $\boldsymbol{K}_a[s]$ be the moment generating matrix of X' killed at rate a while $\widetilde{J}(t) = 0$ (note that then the crossing probabilities coincide with those of the original model). Then, from [6] p. 41, we find that $\boldsymbol{K}_a[s]$ is, in obvious block–partitioned notation, given by

$$\boldsymbol{K}_a[s] = \left(\begin{array}{c|c} \boldsymbol{K}_a^{(1)}[s] & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{K}_a^{(2)}[s] \end{array}\right) + \left(\begin{array}{c|c} \widetilde{\boldsymbol{Q}}_{11} & \widetilde{\boldsymbol{Q}}_{12} \\ \hline \widetilde{\boldsymbol{Q}}_{21} & \widetilde{\boldsymbol{Q}}_{22} \end{array}\right) \tag{44}$$

where

$$\boldsymbol{K}_a^{(i)}[s] \;=\; \begin{pmatrix} -\lambda_i - a + s\mu_i + s^2\sigma_i^2/2 & \lambda_i^{(-)}\boldsymbol{\alpha}_i^{(-)} & \lambda_i^{(+)}\boldsymbol{\alpha}_i^{(+)} \\ \boldsymbol{t}_i^{(-)} & \boldsymbol{T}_i^{(-)} - s\boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{t}_i^{(+)} & \boldsymbol{0} & \boldsymbol{T}_i^{(+)} + s\boldsymbol{I} \end{pmatrix} \tag{45}$$

and $\widetilde{\boldsymbol{Q}}_{ii}$ is the matrix of the size of $\boldsymbol{K}_a^{(i)}$ with $-\eta_i$ on position $(1,1)$ and zeros for the rest. and $\widetilde{\boldsymbol{Q}}_{ij}, i \neq j$ has a everywhere zeros except on $(1,1)$ where it has η_i as entry.

We determine now the eigenstructure of $\boldsymbol{K}_a[s]$. As before we see from (3) that under the model (2), κ_i, the Lévy exponent of X^i, is the ratio between two polynomials of degrees $p_i - \epsilon_i$ and p_i resp. where $\epsilon_i = 2, 1, 0$ if $\sigma_i \neq 0$, $(\sigma_i = 0, \mu_i \neq 0)$ and $(\mu_i = \sigma_i = 0)$, respectively. Hence the equation

$$\eta_1\eta_2 = (\kappa_1(s) - a - \eta_1)(\kappa_2(s) - a - \eta_2) \tag{46}$$

has $p_1 + p_2$ roots which we denote by $\varrho_1, \ldots \varrho_{p_1+p_2}$. For each $r = 1, \ldots, p_1 + p_2$ define

$$\boldsymbol{h}^{(r)} = \begin{pmatrix} \gamma_r \boldsymbol{k}_1^{(r)} \\ -\boldsymbol{k}_2^{(r)} \end{pmatrix} \text{ where } \boldsymbol{k}_i^{(r)} = \begin{pmatrix} 1 \\ \left(\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(-)}\right)^{-1} \boldsymbol{t}_i^{(-)} \\ \left(-\varrho_r \boldsymbol{I} - \boldsymbol{T}_i^{(+)}\right)^{-1} \boldsymbol{t}_i^{(+)} \end{pmatrix} \tag{47}$$

and $\gamma_r = (\kappa_2(\varrho_r) - a - \eta_2)/\eta_2$. By straightforward algebra we can check:

Lemma 3 *For $j = 1, \ldots, p_1 + p_2$, $\boldsymbol{K}_a[\varrho_j]\boldsymbol{h}^{(j)} = \boldsymbol{0}$.*

We adapt now the semi-Markov generalization of the Kella-Whitt martingale introduced by Asmussen and Kella [7]. First, we introduce some more notation.

By Y' we will denote the process X' reflected in its supremum, that is, $Y' = \{Y'_t, t \geq 0\}$ with

$$Y'_t = \sup_{0 \leq s \leq t} X'_s \vee Y'_0 - X'_t.$$

By $L' = \{L'_t, t \geq 0\}$ we will denote the supremum of X', $L'_t = \sup_{s \leq t} X'_s \vee Y'_0$. Finally, we introduce the time spent by Y' in *phase 0* (which is the time of the original regime switching Lévy process) up to time t by

$$T'_0(t) = \int_0^t \mathbf{1}_{\{\widetilde{J}(s)=0\}} \mathrm{d}s.$$

Let $\mathbb{P}_{(i,l),y}$ refer to the case $J_0 = (i,l), Y'_0 = y$ and $\tau' = \tau'_k = \inf\{t > 0 :\ J_0 = j, Y'_t = k\}$. It is immediate by a sample path comparison that $\tau = T'_0(\tau')$ and $\pi^{(i,j)}_{j'} = \mathbb{E}_{(i,0),y}[\mathrm{e}^{-aT'_0(\tau')} \mathbf{1}_{\{J'_{\tau'}=(j,j')\}}]$ for $i, j \in \{1,2\}$ and $j' \in M^{j(-)}$. Finally, let

$$\delta^{(i,j)}_\ell = \mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT'_0(t)} \mathbf{1}_{\{J'_t=(j,\ell)\}} \mathrm{d}L'_t\right], \qquad j \geq 0.$$

By $\mathbf{1}_{J'_t} = \mathbf{1}_{(r,s)}$, we denote a row-vector of the length of $\boldsymbol{K}_a$ with all zeros but a one on a position which corresponds with phase s in regime r.

The theorem below identifies a vector martingale (48), a set of $p_1 + p_2$ scalar martingales (49) and an “optional stopping system” (50).

Theorem 2

(1) The process

$$\mathrm{e}^{-aT'_0(t)+bY'_t} \mathbf{1}_{J'_t} - \mathrm{e}^{bY'_0} \mathbf{1}_{J'_0} + b \int_0^t \mathrm{e}^{-aT'_0(u)} \mathbf{1}_{J'_u} dL'_u - \int_0^t \mathrm{e}^{-aT'_0(u)+bY'_u} \mathbf{1}_{J'_u} du \ \ \boldsymbol{K}_a[-b] \quad (48)$$

is a mean zero (vector) $\mathbb{P}$–martingale.

(2) Let ϱ_r denote any root of the equation (46). Then

$$M_t \ = \mathrm{e}^{-aT'_0(t)-\varrho_r Y'_t} h^{(j)}_{J'_t} - \ \mathrm{e}^{-\varrho_r y} h^{(r)}_{J'_0} - \varrho_r \int_0^t \mathrm{e}^{-aT'_0(s)} h^{(j)}_{J'_s} \mathrm{d}L'_s \qquad (49)$$

are mean zero (scalar) martingales for each $j = 1, \ldots, p_1 + p_2$.

(3) Let $i \in \{1,2\}$ and $y \in [0,k)$. The numbers

$$\pi^{(i,j)}_0, \ldots, \pi^{(i,j)}_{m^{(-)}_j} \ \ \textit{and} \ \ \delta^{(i,j)}_0, \ldots, \delta^{(i,j)}_{m^{(+)}_j} \qquad (j = 1, 2)$$

solve the system of the $p = p_1 + p_2$ linear equations

$$
\begin{aligned}
\mathrm{e}^{-\varrho_1 y} h^{(1)}_{(i,0)} &= \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(-)}} \pi_\ell^{(i,j)} \mathrm{e}^{-\varrho_1 k} h^{(1)}_{j,\ell} - \varrho_1 \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(+)}} \delta_\ell^{(i,j)} h^{(1)}_{j,\ell}, \\
\mathrm{e}^{-\varrho_2 y} h^{(2)}_{(i,0)} &= \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(-)}} \pi_\ell^{(i,j)} \mathrm{e}^{-\varrho_2 k} h^{(2)}_{j,\ell} - \varrho_2 \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(+)}} \delta_\ell^{(i,j)} h^{(2)}_{j,\ell}, \\
&\vdots \\
\mathrm{e}^{-\varrho_p y} h^{(p)}_{(i,0)} &= \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(-)}} \pi_\ell^{(i,j)} \mathrm{e}^{-\varrho_p k} h^{(p)}_{j,\ell} - \varrho_p \sum_{j=1}^{2} \sum_{\ell=0}^{m_j^{(+)}} \delta_\ell^{(i,j)} h^{(p)}_{j,\ell}.
\end{aligned}
\tag{50}
$$

where $h^{(r)}_{j,\ell}$ *is the coordinate of* $\boldsymbol{h}^{(r)}$ *corresponding to regime* j *and phase* ℓ*.*

The proof is provided in Section 5.5.

5 Proofs

5.1 Proof of Theorem 1

We start with a lemma which explores properties of v^*:

Lemma 4 *The function* $v^* : [0,\infty) \to [1,\infty)$ *is convex. If* $v^*(0) > 1$*, then there exists a unique* $k^* \in (0,\infty]$ *such that*

$$
\begin{cases} \exp(x) < v^*(x) < \exp(k^*) & \text{if } 0 \le x < k^*, \\ \exp(x) = v^*(x) & \text{if } k^* < \infty \text{ and } k^* \le x. \end{cases}
$$

If $v^*(0) = 1$*, then* $v^* = \exp$*.*

Proof For τ arbitrary it holds that

$$
\mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau + Y_\tau}] = \mathbb{E}[\mathrm{e}^{-q\tau + \overline{X}_\tau \vee y}] = \mathbb{E}[\mathrm{e}^{\overline{X}_{\eta(q)} \vee y} \mathbf{1}_{\{\tau < \eta(q)\}}], \tag{51}
$$

where $q = a + r$ and $\eta(q)$ is an independent exponential random variable with parameter q. Since $\kappa(1) = r$ and $q/(q - \kappa(1)) = q/a$ is equal to $\varphi_q^+(1) \times \varphi_q^-(1)$, it follows that the expectation on the right-hand side of the previous display is finite uniformly in τ. The assertions follow from the following two observations:

(1) $v^*(x) \ge \mathrm{e}^x$, which follows by choosing $\tau = 0$ in (11);

(2) $x \mapsto v^*(x)$ and $x \mapsto \mathrm{e}^{-x}v^*(x)$ are convex and non-decreasing and non-increasing respectively.

Observation (2) is shown as follows. For each fixed $\tau \in \mathcal{T}$ and ω the functions $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega) - x)$ and $x \mapsto \exp(-a\tau(\omega) + \overline{X}_{\tau(\omega)}(\omega) \vee x - X_{\tau(\omega)}(\omega))$ are convex and non-increasing and non-decreasing respectively. Integration over ω and taking the supremum over τ preserve these properties. QED

Proof of Theorem 1 Let $f_t = \exp(-at + \sup_{0\leq s\leq t} X_s \vee m)$ denote the system of pay-off functions belonging to the problem (8). Note that f_t has no negative jumps and $\{\mathrm{e}^{-r\tau} f_\tau : \tau \in \mathcal{T}\}$ is uniformly integrable with respect to $\mathbb{P}$. Under these conditions, it is straightforward to check that Theorem 2 in Shiryaev et al. [48] continues to hold. (Theorem 2 in [48] is stated in the setting of the standard complete Black-Scholes market, but the completeness plays no role in the proof.) This now implies that the optimal stopping time in (8) is given by

$$
\begin{aligned}
\tau^* &= \inf\{t \geq 0 : \operatorname*{esssup}_{\tau\in\mathcal{T},\tau\geq t} \mathbb{E}[\mathrm{e}^{-r(\tau-t)} f_\tau|\mathcal{F}_t] \leq f_t\} \\
&= \inf\{t \geq 0 : \sup_{\tau\in\mathcal{T}} \mathbb{E}_{X_t,\overline{X}_t\vee m}[\mathrm{e}^{-r\tau} f_\tau] \leq \mathrm{e}^{at} f_t\} \\
&= \inf\{t \geq 0 : V^*(X_t, \overline{X}_t \vee m) = \mathrm{e}^{X_t} v^*(\overline{X}_t \vee m - X_t) \leq \mathrm{e}^{\overline{X}_t\vee m}\}
\end{aligned}
$$

where in the second line, we used the Markov property of $(X_t, \overline{X}_t \vee m)$ and $\mathbb{P}_{x,z}$, $z \geq m$, denotes the probability measure under which the process $(X_t, \overline{X}_t \vee m)$ starts in (x, z). The final line follows by using the $\mathbb{P}$-martingale $M = \{M_t\}_{t\geq 0}$ with $M_t = \exp(X_t - X_0 - rt)$ as equivalent change of measure. The final line of the previous display combined with Lemma 4 implies that the optimal stopping time is a crossing time τ_{k^*} of Y, where the optimal level k^* can be found by optimisation. Since $\tau_k \to \infty$ if k tends to infinity, we deduce from (51) that k^* is finite. Thus, we have for the optimal level

$$
k^* = \inf\left\{k \geq 0 : k \in \arg\max_{k\geq y}[\mathrm{e}^k v_k^{(1)}(y)], \qquad \text{for all } y \geq 0\right\}. \tag{52}
$$

We claim that in (52) 'for all $y \geq 0$' can be replaced by 'for $y = 0$'. This can be seen as follows. By the Markov property of Y and the definition of v_k we find that $v^* = \mathrm{e}^{k^*} v_{k^*}^{(1)}$ satisfies

$$
\mathrm{e}^{-a(\tau(k^*)\wedge t)} v^*(Y_{\tau(k^*)\wedge t}) = \mathbb{E}^{(1)}[\mathrm{e}^{-a\tau(k)} v^*(Y_{\tau(k^*)})|\mathcal{F}_t],
$$

for all $t \geq 0$. It follows that $\{\mathrm{e}^{-a(\tau\wedge t)} v^*(Y_{\tau\wedge t})\}_{t\geq 0}$ is a $\mathbb{P}^{(1)}$-martingale for $\tau = \tau(k^*)$. In particular, setting $\tau = \tau(k^*) \wedge \tau(\ell)$, Doob's optional stopping theorem yields that v^* satisfies for all $\ell > 0$ and $y \in [0, k^* \wedge \ell]$

$$\begin{aligned} v^*(y) &= v^*(0)\mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau_0<\tau\}}] + \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau}v^*(Y_\tau)\mathbf{1}_{\{\tau_0>\tau\}}] \\ &= c + (\mathcal{G}v^*)(y), \end{aligned} \tag{53}$$

where $c = v^*(0)\mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau_0<\tau\}}]$ and $\mathcal{G} : B \to B$ the operator given by $v \mapsto \mathbb{E}_y^{(1)}[\mathrm{e}^{-a\tau}v(Y_\tau)\mathbf{1}_{\{\tau_0>\tau\}}]$ (where B is the Banach space of the bounded functions on $[0, k^* \wedge \ell]$ with the supremum norm). By the contraction theorem, we find that v^* is the unique solution in B of $v = c + \mathcal{G}v$. Thus, if $k \le k^*$ is such that $\mathrm{e}^k v_k^{(1)}(0) = v^*(0)$ then $\mathrm{e}^k v_k^{(1)} = v^*$ and the claim is proved.

Finally, we show that the argmax in (52) is a singleton. Suppose that there exists a $k_* > k^*$ in the argmax in (52). Then this would imply that $\mathrm{e}^{k^*} v_{k^*}^{(1)} = \mathrm{e}^{k_*} v_{k_*}^{(1)}$ and that $\{\mathrm{e}^{-a(t\wedge\tau')+Y_{t\wedge\tau'}}\}_{t\ge0}$ with $\tau' = \tau(k^*) \wedge \tau(k_*)$ would be a $\mathbb{P}^{(1)}$-martingale. Since $Y_{t\wedge\tau'}$ has the same law as $-X_{t\wedge T'}$, with T' the first time $-X$ exits (k^*, k_*), and $\{\mathrm{e}^{+r(t\wedge T')-X_{t\wedge T'}}\}_{t\ge0}$ is a $\mathbb{P}^{(1)}$-martingale, we reach a contradiction. Thus $k^* = k_*$ and the proof is finished. QED

5.2 Proof of linear independence

Here we show that the vectors $\hat{\boldsymbol{f}}^{(-)}[\rho_i], i \in \mathcal{I}^{(-)}$ are linearly independent. We assume that the roots ρ_i with $i \in \mathcal{I}^{(-)}$ are distinct and that the representation $(m^{(-)}, \boldsymbol{T}^{(-)}, \boldsymbol{\alpha}^{(-)})$ is minimal. In particular, this means that $\boldsymbol{T}^{(-)}$ has no eigenvalues with multiple geometric multiplicity. We distinguish between the cases that $X^{(+)}$ is a subordinator or not.

• $X^{(+)}$ is a subordinator. Note that in this case $\#\mathcal{I}^{(-)} = m^{(-)}$. Writing $\boldsymbol{C}$ for the matrix of (generalised) eigenvectors of $\boldsymbol{T}^{(-)}$ and $\boldsymbol{J} = \boldsymbol{C}^{-1}\boldsymbol{T}^{(-)}\boldsymbol{C}$ for its Jordan normal form, we have to show the linear independence of the vectors $C(\rho_i\boldsymbol{I} - \boldsymbol{J})^{-1}\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$ for $i \in \mathcal{I}^{(-)}$. We claim that this linear independence is equivalent with invertibility of the matrix $\boldsymbol{M}$ with rows $\sum_{k=1}^{j-1} m^{(-,k)} + 1$ till $\sum_{k=1}^{j} m^{(-,k)}$ given by

$$\left(\tfrac{\rho_i}{(\rho_i - \eta^{(-,j)})^\ell}, i \in \mathcal{I}^{(-)}\right), \qquad \ell = 1, \ldots, m^{(-,j)} \tag{54}$$

where $\eta_j^{(-)}$ are the eigenvalues of $\boldsymbol{T}^{(-)}$ with multiplicitites $m^{(-,j)}$. The claim follows by linear algebra. Indeed, denoting by $\boldsymbol{v}^{(j,m)}$ the column of $\boldsymbol{C}$ that lies in the kernel of $(\boldsymbol{J} - \eta_j^{(-)}\boldsymbol{I})^m$, but not in the kernel of $(\boldsymbol{J} - \eta_j^{(-)}\boldsymbol{I})^{m-1}$, we see that $\mathbf{1}$ is not in the span of $\{\boldsymbol{v}^{(j,m)}\}_{j,m<m^{(-,j)}}$ (For suppose this were the case, then applying $\prod_j(\boldsymbol{J} - \eta_j^{(-)}\boldsymbol{I})^{m_j}$, where $m_j = m^{(-,j)} - 1$, to the vector $\mathbf{1}$ would lead to a contradiction, since we assumed that $\boldsymbol{T}^{(-)}$ has no eigenvalues of multiple geometric multiplicity). This implies that the vector $\boldsymbol{C}^{-1}\mathbf{1}$ (and hence $\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$ as $\boldsymbol{T}^{(-)}$ is negative definite) is non-zero in all coordinates corresponding to

the eigenvectors $\boldsymbol{v}^{(j,m^{(-,j)})}$. Recalling the form of the inverse $(\lambda \boldsymbol{I} - \boldsymbol{J})^{-1}$ for the Jordan form $\boldsymbol{J}$, it follows by writing out the equations that the vectors $C(\rho_i \boldsymbol{I} - \boldsymbol{J})^{-1}\boldsymbol{J}\boldsymbol{C}^{-1}\mathbf{1}$ are linearly independent if and only if $\boldsymbol{M}$ is one-to-one and the claim follows.

Next we show that $\boldsymbol{M}$ is invertible. Consider now the system $\boldsymbol{M}\boldsymbol{c} = -\boldsymbol{v}$, where $\boldsymbol{v}$ is the vector with $\varphi_a^-(\infty)$ in coordinates $1, m^{(-,1)} + 1, m^{(-,2)} + 1, \ldots$ and the rest zeros. Recall we restricted ourselves to the cases where the roots of $\kappa(s) = a$ with negative real part are distinct and not in the spectrum of $\boldsymbol{T}^{(-)}$. Then we can check that any solution $\boldsymbol{c}$ of this system gives rise to a partial fraction decomposition of $\varphi_a^-(s) - \varphi_a^-(\infty)$. Indeed, recall that we can write $\varphi_a^-(s) = p(s)/q(s)$ for polynomials p, q of degree $m^{(-)}$. Taking $\boldsymbol{c}$ to be a solution of above system we have that

$$p(s) = \left(\sum_{i \in \mathcal{I}^{(-)}} c_i \rho_i / (\rho_i - s) + \varphi_a^-(\infty) \right) q(s) \tag{55}$$

since both sides of the equation are polynomials of the same degree, any root of the left-hand side is also a root of the right-hand side with the same multiplicity and $(p/q)(\infty) = \varphi_a^-(\infty) > 0$. By unicity of this partial fraction decomposition, we deduce that the square matrix $\boldsymbol{M}$ is invertible.

• $X^{(+)}$ is not a subordinator. Note that linear independence of the $m^{(-)} + 1$ vectors $\hat{\boldsymbol{f}}^{(-)}[\rho_i]$ of length $m^{(-)} + 1$ for $i \in \mathcal{I}^{(-)}$ is equivalent to the system

$$0 = \sum \alpha_i = \sum \alpha_i (\rho_i \boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1} \boldsymbol{t}^{(-)},$$

where the sums is over $i \in \mathcal{I}^{(-)}$, having only the trivial solution. By the same argument as above it follows that this system can be equivalently reformulated as $\widetilde{\boldsymbol{M}}\boldsymbol{\alpha} = \mathbf{0}$, where $\widetilde{\boldsymbol{M}}$ is the matrix with the final $m^{(-)}$ rows given by (54) and the first row of ones. So we are done if we prove that $\widetilde{\boldsymbol{M}}$ is invertible. We write $\varphi_a^-(s) = p(s)/q(s)$, where p, q are polynomials of degree $m^{(-)}$ and $m^{(-)}+1$, respectively. As above it is enough show that any solution of $\widetilde{\boldsymbol{M}}\widetilde{c} = \mathbf{1}_1$ corresponds to a partial fraction decomposition of ϕ_a^-, since this decomposition is unique. Let $\widetilde{c}$ be any solution of $\widetilde{\boldsymbol{M}}\widetilde{c} = \mathbf{1}_1$. Then

$$q(s) \sum_i \widetilde{c}_i \rho_i / (\rho_i - s) \tag{56}$$

is a polynomial of (at most) degree $m^{(-)}$ and any root of $p(s) = 0$ is also a root of this polynomial. Since $\varphi_a^-(0) = 1 = \sum_i \widetilde{c}_i$, the polynomial (56) is equal to p. Hence $\widetilde{c}$ gives indeed rise to a partial fraction decomposition of φ_a^-.

Spectral proof of Lemma 1(2) First suppose that the roots ρ_i, $i \in \mathcal{I}^{(-)}$, are distinct. Define the function $\widetilde{u} : \mathbf{R} \to \mathbf{R}$ by setting $\widetilde{u}(x)$ for $x \geq 0$ equal to the right-hand side of (19) and equal to 1 for $x < 0$. Since, if the roots ρ_i of (13) with negative real part are distinct, A_i^- is the coefficient of $\rho_i/(\rho_i - s)$ in the partial fraction decomposition of $\varphi_a^-(s) - \varphi_a^-(\infty)$, we find $\sum_{i\in\mathcal{I}^{(-)}} A_i^- = 1 - \varphi_a^-(\infty)$. In particular, we find that $u(0) = 1$ iff $X^{(+)}$ is not a subordinator. (using from Lemma 1(1) that $\#\mathcal{I}^{(-)} > \#\mathcal{J}^{(-)}$ iff $X^{(+)}$ is not a subordinator). Moreover, the Cayley-Hamilton theorem implies that we have the following matrix identity

$$-\varphi_a^-(\infty)\boldsymbol{I} = \textstyle\sum_{i\in\mathcal{I}^{(-)}} A_i^- \rho_i(\rho_i \boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}. \tag{57}$$

Using the foregoing identities, it is straightforward to check that $\widetilde{u}$ satisfies for $z > 0$

$$\begin{aligned}(\Gamma\widetilde{u})(z) - a\widetilde{u}(z) &= \sum_i A_i^- \mathrm{e}^{\rho_i z}(\kappa(\rho_i) - a) \\ &+ \lambda^{(-)}\boldsymbol{\alpha}^{(-)}\left[(-\boldsymbol{T}^{(-)})^{-1} - \sum_i A_i(\rho_i\boldsymbol{I} - \boldsymbol{T}^{(-)})^{-1}\right]\mathrm{e}^{\boldsymbol{T}^{(-)}z}\boldsymbol{t}^{(-)} = 0\end{aligned} \tag{58}$$

where Γ is the infinitesimal generator of X. By applying Itô's lemma to $\mathrm{e}^{-at}\widetilde{u}(X_t)$ restricted to $\{t < T(0)\}$, we then see that $\exp(-a(t\wedge T(0)))\widetilde{u}(X_{t\wedge T(0)})$ is a martingale. The proof is completed by the bounded convergence theorem:

$$\widetilde{u}(z) = \lim_{t\to\infty} \mathbb{E}_z[\mathrm{e}^{-a(t\wedge T(0))}\widetilde{u}(X_{t\wedge T(0)})] = \mathbb{E}_z[\mathrm{e}^{-aT(0)}] = \mathbb{P}(-I_a > z),$$

where we used that $u(z) = 1$ for $z < 0$ and $u(0) = 1$ iff 0 is regular for $(-\infty, 0)$ for $X^{(+)}$ which is the case iff $X^{(+)}$ is not a subordinator. The case of multiple roots follows from the single root case by approximation. QED

Proof of Proposition 4 Analogous as in Section 5.2, we find that the assumed linear independence implies that the system (34)–(35) has a unique solution. Define the function $\widetilde{v}$ on $[0,\infty)$ by the right-hand side of (36) for $y \leq k$ and by $\exp(b(y-k))$ for $y > k$. From the explicit form of $\widetilde{v}$ we straightforwardly check that $\widetilde{v}'(0^+) = 0$ (if $\sigma \neq 0$ or $\mu > 0$) and $\Gamma'\widetilde{v}(x) = a\widetilde{v}(x)$ for $x \in (0,k)$ where Γ' acts on $f \in C^2(0,k)$ as

$$\begin{aligned}\Gamma' f(x) = \frac{\sigma^2}{2} f''(x) - \mu\, f'(x) &+ \lambda^{(-)}\int_0^\infty \Big(f(x+z) - f(x)\Big)F^{(-)}(\mathrm{d}z) \\ &+ \lambda^{(+)}\int_0^\infty \Big(f((x-z)^+) - f(x)\Big)F^{(+)}(\mathrm{d}z),\end{aligned} \tag{59}$$

for $x \in (0,k)$. Applying then Itô's lemma to $\exp(-at)\widetilde{v}(Y_t)$ on the set $\{t \leq \tau_k\}$ and using the two foregoing properties of $\widetilde{v}$, it follows that $\{\exp(-a(t \wedge$

$\tau_k)\widetilde{v}(Y_{t\wedge\tau_k}), t \geq 0\}$ is a martingale. Thus, by bounded convergence combined with the fact that $\widetilde{v}(y) = \exp(b(y-k))$ for $y > k$ and $\widetilde{v}(k^-) = 1$ if $\sigma \neq 0$ or $\mu < 0$, we deduce

$$\begin{aligned}\widetilde{v}(y) &= \lim_{t\to\infty} \mathbb{E}_y[\mathrm{e}^{-a(\tau_k\wedge t)}\widetilde{v}(Y_{\tau_k\wedge t})] \\ &= \mathbb{E}_y[\mathrm{e}^{-a\tau_k+b(Y_{\tau_k}-k)}] = v_k(y),\end{aligned}$$

which completes the proof. QED

5.4 *Proof of Lemma 2*

(1)-(2) Recall v_k satisfies $\Gamma' v_k = a v_k$, where Γ' is given in (59). In the case $\mu \geq \sigma = 0$, one finds that $v_k(0) = \frac{\lambda^{(-)}}{\lambda^{(-)}+a}\int_0^\infty v_k(x)F^{(-)}(\mathrm{d}x)$. Taking then the limit of $k \downarrow 0$ and using that $v_k(x) = \mathrm{e}^{x-k}$ for $x \geq k$, one find that $\lim_{k\downarrow 0} v_k(0) = \iota$. In the case $\mu < \sigma = 0$, we take the limit of $x \uparrow k$ to find that $\Gamma' v_k(k^-) = a v_k(k^-)$ which reads as $-\mu v_k'(k^-) + \lambda^{(+)}\int_0^\infty v_k((k-x)^+)F^{(+)}(\mathrm{d}x) = (a+\lambda^{(+)})v_k(k^-)$. Letting then k tend to zero and using that $v_k(k^-) = 1$ and $\lim_{k\downarrow 0} v_k(0) = 1$ shows that $v_k'(k^-) \to -a/\mu$.
(3) Denote by $\tau' = \tau'(\epsilon) = \inf\{t \geq 0 : Y_t < k - \epsilon\}$, write $\tau = \tau(k)$ and $k(\epsilon) = k - \epsilon$ and let ς be the first jump time of Y. The idea is now to exploit the fact that on $\{t < \varsigma\}$ the process X is in law equal to a Brownian motion with drift. Splitting the probability space according to whether $\tau < \varsigma$ and using the Markov property of Y we find that

$$\begin{aligned}\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\varsigma\}}] - 1 &= v_k(k(\epsilon))\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}] + \mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\tau'\wedge\varsigma\}}] - 1 \\ &= (v_k(k(\epsilon)) - 1)\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}] \\ &\quad + \mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a(\tau\wedge\tau')}\mathbf{1}_{\{\tau\wedge\tau'<\varsigma\}}] - 1, \qquad (60)\end{aligned}$$

where we used that $Y_\tau = k$ on $\{\tau < \varsigma\}$. Since the jump component is independent of the rest of the process and have total jump rate λ, we find, invoking Proposition 3, that

$$w_k(x) := \mathbb{E}_x[\mathrm{e}^{-a\tau}\mathbf{1}_{\{\tau<\varsigma\}}] = \tilde{s}(x)/\tilde{s}(k),$$

where $\tilde{s}(x) = \tilde{\rho}_2\mathrm{e}^{-\tilde{\rho}_1 x} - \tilde{\rho}_1\mathrm{e}^{-\tilde{\rho}_2 x}$ with $\tilde{\rho}_1 < 0 < \tilde{\rho}_2$ the roots of $\frac{\sigma^2}{2}s^2 + \mu s = a + \lambda$ (where the parameters are the ones under $\mathbb{P}^{(1)}$). Recalling that $\kappa_1(-1) = -r$ we find that $\frac{\sigma^2}{2} - m - \lambda \leq -r$ and thus $\tilde{\rho}_1 < -1$. It is then a matter of algebra to verify that the function $k \mapsto w_k'(k^-)$ is increasing (with positive derivative for $k > 0$) and converges to 0 and $-\tilde{\rho}_1 > 1$ as $k \downarrow 0$ and $k \to \infty$ respectively. Similarly, by a Wald-martingale argument as in the proof of Proposition 2, we find that

$$t_k(x) := \mathbb{E}_x[\mathrm{e}^{-a(\tau\wedge\tau')}\mathbf{1}_{\{\tau\wedge\tau'<\varsigma\}}]$$
$$= \mathrm{e}^{\tilde{\rho}_1(k-x)}A_1 - \mathrm{e}^{\tilde{\rho}_2(k-x)}A_2,$$

where $A_i = \frac{\mathrm{e}^{\tilde{\rho}_i\epsilon}-1}{\mathrm{e}^{\tilde{\rho}_2\epsilon}-\mathrm{e}^{\tilde{\rho}_1\epsilon}}$. After some computation, it follows that $(t_k(k-\frac{\epsilon}{2})-1)/\epsilon \to 0$ as $\epsilon \downarrow 0$. Similarly, one verifies that $\mathbb{E}_{k(\frac{\epsilon}{2})}[\mathrm{e}^{-a\tau'}\mathbf{1}_{\{\tau'<\tau<\varsigma\}}] \to 1/2$ as $\epsilon \downarrow 0$. Dividing then the left- and right-hand side of (60) by $\epsilon/2$ and letting ϵ go to zero we find that $w_k'(k^-) = v_k(k^-)$, which establishes (iii). QED

5.5 Proof of Theorem 2

Let the process $Z = \{Z_t, t \geq 0\}$ be given by

$$Z_t = Y_t' - \frac{a}{b}T_0'(t) = -X_t' + L_t' - \frac{a}{b}T_0'(t).$$

Since Z has continuous sample paths, applying Theorem 2.1 d) of [7]), we find that – without restrictions on b, $M = \{M_t, t \geq 0\}$ with

$$\begin{aligned} M_t = &\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_0[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}L_s' \\ &- a\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s=0)\mathrm{d}s \\ = &\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{K}_a[-b] + \mathrm{e}^{by}\mathbf{1}_{J_0} - \mathrm{e}^{-bZ_t}\mathbf{1}_{J_t} + b\int_0^t \mathrm{e}^{-aT_0'(s)}\mathbf{1}_{J_s}\mathrm{d}L_s', \end{aligned}$$

is a zero mean $\mathbb{P}_{0,y}$ (row) martingale. We used that L_t' can increase only if X_t' is equal to its current supremum or $Y_t' = 0$. Moreover $\int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}I(J_s=0)\mathrm{d}s = \int_0^t \mathrm{e}^{bZ_s}\mathbf{1}_{J_s}\mathrm{d}s\boldsymbol{\Delta}$ with $\boldsymbol{\Delta}$ a diagonal matrix with a 1 on positions 1 and p_1+1 and the rest zeros. Choosing $-b$ to be a root of $\kappa(s) = a$ and multiplying by the zero-eigenvectors of $K_a[-b]$ (using Lemma 3) completes the proof of 1 and 2.

Since $M_{t\wedge\tau'}$ is bounded for all t, for each j, can we apply optional stopping theorem to M at $\tau' = \tau_k'$, i.e. $\mathbb{E}_{(i,0),y}[M_{\tau'}] = \mathbb{E}_{(i,0),y}[M_0] = 0$. Since $\sup_{s\leq t} X_s'$ can increase only when $Y_t' = 0$ and $J_t \geq 0$, we find

$$\mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT_0'(s)}h_{J_s}^{(r)}\mathrm{d}L_s'\right] = \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}} h_{j,\ell}^{(r)}\mathbb{E}_{(i,0),y}\left[\int_0^{\tau'} \mathrm{e}^{-aT_0'(s)}I(J_s=(j,\ell))\,dL_s'\right]$$

which is equal to $\sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(+)}}\delta_k^{(i,j)}h_{j,\ell}^{(r)}$. Similarly, we must have $J_{\tau'} \le 0$ so that

$$\begin{aligned}\mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r Z_{\tau'}}h_{J_{\tau'}}^{(r)}\right] &= \mathbb{E}_{(i,0),y}\left[\mathrm{e}^{-\varrho_r k - aT_0'(\tau')}h_{J_{\tau'}}^{(r)}\right] \\ &= \sum_{j=1}^{2}\sum_{\ell=0}^{m_j^{(-)}}\pi_\ell^{(i,j)}\mathrm{e}^{-\varrho_r k}h_{j,\ell}^{(r)}.\end{aligned}$$

Thus the rth equation is the same as $\mathbb{E}_{0,y}[M_{\tau'}] = 0$. If the roots ϱ_r are different, the equations are linearly independent, which can be proved as in Section 5.2. QED

Appendix

A Exponential tilting of X

Consider the probability measure $\mathbb{P}^{(u)}$ given by $\mathbb{P}^{(u)}(A) = \mathbb{E}[\mathrm{e}^{uX_t - t\kappa(u)}; A]$, $A \in \mathcal{F}_t$. It is standard (e.g. [6] p. 38) that X is again a Lévy process w.r.t. $\mathbb{P}_s$, with Lévy exponent given by $\kappa_u(s) = \kappa(u+s) - \kappa(u)$ corresponding to the following change of parameters:

$\mathbb{P}$	μ	σ^2	$\lambda^{(+)}$	$F^{(+)}$	$\lambda^{(-)}$	$F^{(-)}$
$\mathbb{P}^{(u)}$	$\mu + u\sigma^2$	σ^2	$\lambda^{(+)}\hat{F}^{(+)}[-u]$	$F_u^{(+)}$	$\lambda^{(-)}\hat{F}^{(-)}[u]$	$F_{-u}^{(-)}$

where $F_u^{(+)}(dx) = \mathrm{e}^{ux}F^{(+)}(dx)/\hat{F}^{(+)}[-u]$, $F_{-u}^{(-)}(dx) = \mathrm{e}^{-ux}F^{(-)}(dx)/\hat{F}^{(-)}[u]$. These distributions are again phase–type, as follows by the following result from [1]:

Lemma 5 *Let F be phase–type with parameters $(\boldsymbol{\alpha}, \boldsymbol{T})$ and let $F_u(dx) = \mathrm{e}^{ux}F(dx)/\hat{F}[-u]$. Define $\boldsymbol{k} = (-u\boldsymbol{I} - \boldsymbol{T})^{-1}\boldsymbol{t}$ and let $\boldsymbol{\Delta}$ be the diagonal matrix with the k_i on the diagonal. Then F_u is phase–type with parameters*

$$\boldsymbol{\alpha}_u = \boldsymbol{\alpha}\boldsymbol{\Delta}/\hat{F}^{(+)}[-u], \quad \boldsymbol{T}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{T}\boldsymbol{\Delta} + u\boldsymbol{I}.$$

Further, $\boldsymbol{t}_u = \boldsymbol{\Delta}^{-1}\boldsymbol{t}$.

References

[1] Asmussen, S.: Exponential families generated by phase–type distributions and other Markov life times. Scand. J. Statist. **16**, 319–334 (1989)

[2] Asmussen, S.: Phase–type representations in random walk and queueing problems. Ann. Probab. **20**, 772–789 (1992)

[3] Asmussen, S.: Stationary distributions via first passage times. Advances in Queueing (J. Dshalalow, ed.), 79–102 CRC Press (1995)

[4] Asmussen, S. Nerman, O. and Olsson, M.: Fitting phase–type distributions via the EM algorithm. Scand. J. Statist. **23**, 419–441 (1996). Software can be downloaded from `www.maths.lth.se/matstat/staff/asmus`.

[5] Asmussen, S.: Matrix–analytic models and their analysis. Scand. J. Statist. **27**, 193–226 (2000)

[6] Asmussen, S.: Ruin Probabilities, World Scientific 2000

[7] Asmussen, S. and Kella, O.: A multi–dimensional martingale for Markov additive processes and its applications, Adv. Appl. Probab. **32**, 376–393 (2000)

[8] Asmussen, S., Avram, F. and Usabel, M.: Erlangian approximations for finite-horizon ruin probabilities. Astin Bull. **32**, 267–281 (2002)

[9] Avram, F., Chan, T. and Usabel, M.: On the valuation of constant barrier options under spectrally one-sided exponential Lévy models and Carr's approximation for American puts. Stochastic Process. Appl. **100**, 75–107 (2002)

[10] Avram, F., Kyprianou, A.E. and Pistorius, M.R.: Exit Problems for Spectrally Negative Lévy Processes and Applications to (Canadized) Russian Options, to appear in Ann. Appl. Prob.

[11] Barndorff-Nielssen, O.: Processes of normal inverse Gaussian type, Finance Stoch. **2**, 41–68 (2000)

[12] Bertoin, J. *Lévy processes*, Cambridge University Press, 1996.

[13] Bingham, N. H.: Fluctuation theory in continuous time. Adv. Appl. Probab. **7** 705–766 (1975)

[14] Bladt, M., Gonzalez A. and Lauritzen, S.L.: The estimation of phase–type related functionals using Markov chain Monte Carlo methods, Scand. Act. J. (to appear) (2002)

[15] Boyarchenko, S.I., Levendorskiĭ: Option pricing for truncated Lévy processes, Int.Journ.Theor. and Appl. Finance **3**, 549–552 (2000)

[16] Carr, P.: Randomization and the American put, Review of Financial Studies **11**, 597–626 (1998)

[17] Carr, P., Faguet, D.: Fast accurate valuation of American options, Working paper, Cornell University (1994)

[18] Carr, P., H.Geman, H., Madan, D.P. and Yor, M.: The fine structure of asset returns, forthcoming in Journal of Business (1999)

[19] Chan, T.: Pricing contingent claims on stocks driven by Lévy processes, Ann. Appl. Probab. **9**, 504–528 (1998)

[20] Cont, R., Potters, M., Bouchaud, J.-P.: Scaling in stock market data: stable laws and beyond, in Scale invariance and beyond, B. Dubrulle et al., eds., Springer, Berlin 75–85, 1997

[21] Darling, D. A., Liggett, T. and Taylor, H. M.: Optimal stopping for partial sums. Ann. Math. Statist. **43**, 1363–1368 (1972)

[22] Duffie, D. and Harrison, J.M.: Arbitrage pricing of Russian options, Ann. Appl. Probab. **3**, 641–652 (1993)

[23] Eberlein, E. and Keller, U: Hyperbolic distributions in finance. Bernoulli **1**, 281–299 (1995)

[24] Fakeev, A.G.: Optimal stopping of a Markov process, Theory Probab. Appl. **16**, 694–696 (1971)

[25] Fama, E.F.: Mandelbrot and the stable Paretian hypothesis, Journal of Business **36**, 420–429 (1963)

[26] Fama, E.F.: Portfolio analysis in a stable paretian market, Management Science **11**, 404–419 (1965)

[27] Geman, H., Madan D. and Yor, M.: Asset prices are Brownian motion: Only in business time. Preprint, University of Maryland (1999)

[28] Gerber, H. and Shiu, E.: Option pricing by Esscher transforms. Trans. Soc. Actuaries **X LVI**, 99–139 (1994)

[29] Graversen, S.E. and Peškir, G.: On the Russian option: the expected waiting time, Th. Probab. Appl. **24**, 416–425 (1997)

[30] Guo, X.: An explicit solution to an optimal stopping problem with regime switching, J. Appl. Prob. **38**, 464–481 (2001)

[31] Jacod, J. and Shiryaev, A.N. (1987). *Limit theorems for stochastic processes*, Springer Verlag.

[32] Kella, O. and Whitt, W.: Useful martingales for stochastic storage with Lévy input, J. Appl. Prob. **29**, 396–403 (1992)

[33] Kemperman, J.H.B., The passage problem for a stationary Markov chain, The University of Chicago Press, Chicago, 1961

[34] Kou, S.: Pricing options with jump-diffusion models, Preprint Columbia University (2000)

[35] Koponen, I.: Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process, Phys.rev. E. **52**, 1197–1199

[36] Kyprianou, A.E. and Pistorius, M.R.: Perpetual options through fluctuation theory, to appear in Ann. Appl. Prob.

[37] Madan, D.P., Carr, P. and Chang, E.: The variance gamma process and option pricing model, European Finance Review **2**, 79–105 (1998)

[38] Mandelbrot, B.: New methods in statistical economics, Journal of Political Economy **61**, 421–440 (1963)

[39] Mandelbrot, B.: The variation of certain speculative prices, Journal of Business **36**, 394–419 (1963)

[40] Mordecki, E.: Optimal stopping and perpetual options for Lévy processes, Finance and Stochastics **6**, 473–493 (2002)

[41] Mordecki, E.: Ruin probabilities for Lévy processes with mixed-exponential negative jumps, Preprint (2001), forthcoming in Theory of Probability and its Applications.

[42] Mordecki, E. and Moreira, W.: Russian options for a diffusion with negative jumps, Preprint (2001)

[43] Neuts, M.: Matrix–geometric solutions in stochastic models, Johns Hopkins University Press 1981

[44] Peškir, G.; Shiryaev, A. N.: Sequential testing problems for Poisson processes. Ann. Statist. **28**, 837–859 (2000)

[45] Ross, S.M.: Approximating transition probabilities in continuous time Markov Chains. Prob. Eng. Inf. Sci. **3**, 251–264 (1987).

[46] Shepp, L. and Shiryaev, A.N.: The Russian option: reduced regret, Ann. Appl. Prob. **3**, 603–631 (1993)

[47] Shepp, L. and Shiryaev, A.N.: A new look at the pricing of the Russian option, Th. Prob. Appl. **39**, 103–120 (1994)

[48] Shiryaev, A.N., Kabanov, Yu., Kramkov, D.O., and Melnikov, A.V.: Towards the theory of options of both American and European types II. Continuous time., Th. Prob. Appl. **39**, 61–102 (1994)