NECESSARY AND SUFFICIENT CONDITIONS FOR THE POSITIVITY OF SOLUTIONS OF SYSTEMS OF STOCHASTIC PDES

by

Jacky Cresson[1,2], Messoud Efendiev[3] & Stefanie Sonner[3]

1. Laboratoire de Mathématiques Appliquées de Pau, Université de Pau et des Pays de l'Adour, avenue de l'Université, BP 1155, 64013 Pau Cedex, France
2. Institut de Mécanique Céleste et de Calcul des Éphémérides, Observatoire de Paris, 77 avenue Denfert-Rochereau, 75014 Paris, France
3. Helmholtz Zentrum München, Institut für Biomathematik und Biometrie, Ingolstädter Landstrasse 1, D-85764 Neuherberg, Germany

Abstract. — We derive necessary and sufficient conditions for preserving the positive cone for a class of semi-linear parabolic systems of stochastic partial differential equations. These conditions are valid for both Itô's and Stratonovich's interpretation of stochastic differential equations.

Contents

2000 ***Mathematics Subject Classification.*** — 60H15, 35K60, 47H07, 37L30, 65M60, 35K57, 92C15, 35B05, 92B99.

Key words and phrases. — Necessary and sufficient conditions, Positive cone, Semi-linear parabolic systems, Stochastic PDEs.

INTRODUCTION

In this article we study the qualitative behaviour of solutions of systems of semi-linear parabolic equations under stochastic perturbations, in particular the positivity of solutions and the validity of comparison principles. Our results are valid for both Itô's and Stratonovich's interpretation (see [**6**]) of stochastic PDEs.

1. Additive Versus Multiplicative Noise and Itô's Versus Stratonovitch's Interpretation

We first present two simple examples in order to motivate our results. Let us consider the following ordinary differential equation for a real-valued function $u : \mathbb{R} \to \mathbb{R}$

$$\begin{cases} \frac{du}{dt} &= 0 \\ u(0) = u_0, \end{cases} \tag{1}$$

where $u_0 \in \mathbb{R}$. Certainly, this system preserves the positive cone. Indeed, if the initial data satisfies $u_0 \geq 0$, then the corresponding solution remains non-negative $u(t; u_0) = u_0 \geq 0$ for all $t > 0$.

However, if the system is perturbed by an additive noise, that is a white noise modelled by a standard Wiener process $\{W_t, t \geq 0\}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$

$$\begin{cases} du &= 0\, dt + dW_t \\ u(0) = u_0, \end{cases} \tag{2}$$

the positivity is not preserved by the solutions of the perturbed stochastic system (2).

Proposition 1. — *We assume that the initial data satisfies $u_0 \geq 0$. Then, there exists $t^* > 0$ such that the solution u of system (2) becomes negative, that is $u(t^*, \omega; u_0) < 0$.*

Proof. — First note that in case of an additive noise Itô's and Stratonovich's interpretation of the stochastic differential equation (2) lead to the same integral equation (see [**6**] p.28), namely

$$u(t) = u(0) + \int_0^t dW_s = u(0) + W(t) - W(0) = u(0) + W_t,$$

where $\{W_t, t \geq 0\}$ is a standard Wiener process satisfying $W(0) = 0$.

By the law of iterated logarithm holds

$$\liminf_{t\to\infty} \frac{W_t(\omega)}{\sqrt{2t \log\log t}} = -1$$

almost surely (cf.[**5**], p.112). This implies that there is an increasing sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\liminf_{n\to\infty} \frac{W_{t_n}(\omega)}{\sqrt{2t_n \log\log t_n}} = -1$$

almost surely. Consequently, for N_0 sufficiently large follows

$$W_{t_n}(\omega) < -\frac{1}{2}\sqrt{2t_n \log\log t_n}$$

for all $n \geq N_0$. This proves that $W_{t_n} \to -\infty$ when n tends to infinity, which implies that the solution $u(t_n, \omega; u_0) < 0$ if t_n is sufficiently large. □

Instead of an additive noise let us consider the perturbation of the original system by a linear, multiplicative noise of the form

$$\begin{cases} du = 0\,dt + \alpha u \circ dW_t \\ u(0) = u_0, \end{cases} \tag{3}$$

where the constant $\alpha \in \mathbb{R}$. In order to simplify computations we first use Stratonovich's interpretation of the stochastic differential equation as in this case ordinary chain rule formulas apply under a change of variables. The solution of system (3) is explicitly given by the process

$$u(t, \omega; u_0) = u_0 e^{\alpha W_t(\omega)}.$$

Hence, independent of the sign of $\alpha \in \mathbb{R}$, the solutions of the perturbed system preserve positivity. We claim that, if the stochastic differential equation (3) is interpreted in the sense of Itô, the solutions possess the same property.

Proposition 2. — *Independent of the choice of Itô's or Stratonovich's interpretation the solutions of the stochastic problem (3) preserve positivity.*

Proof. — The case of Stratonovich's interpretation has been considered above. There is an explicit formula relating the integral equations obtained through Itô's, respectively Stratonovich's interpretation. Interpreting the stochastic equation (3) in the sense of Itô

$$du = 0\,dt + \alpha u \cdot dW_t,$$

it is equivalent to the following Stratonovich equation

$$du = (0 - \frac{\alpha^2}{2}u)dt + \alpha u \circ dW_t,$$

which can be solved explicitly. Indeed, the transformation $v(t, \omega) := e^{-\alpha W_t(\omega)} u(t, \omega)$ leads to the equation

$$dv = (-\frac{\alpha^2}{2}v)dt,$$

with initial data $v(0) = u_0$. Its solution is the process $v(\omega, t) = u_0 e^{-\frac{\alpha^2}{2}t}$ and consequently,

$$u(\omega, t) = u_0 e^{-(\frac{\alpha^2}{2}t - \alpha W_t(\omega))},$$

which is certainly non-negative if the initial data u_0 is non-negative. We conclude that system (3) preserves the positivity of solutions in Itô's as well as in Stratonovich's interpretation. □

2. Positivity and Stochastic Perturbations of Semi-linear Parabolic Systems: The Simplest Case

We are interested in stochastic perturbations of systems of semi-linear parabolic equations. Our main result applied to scalar equations resembles the well-known fact, which was illustrated by our first example: Additive noise destroys the positivity property of solutions while the positivity of solutions is preserved under perturbations by a linear, multiplicative noise.

As one of the simplest cases of the class of stochastic PDEs we study in this article we now discuss the stochastic perturbation of a system of two semi-linear reaction-diffusion equations. For the reasons pointed out above we consider perturbations by a linear, multiplicative noise in both equations. Let $\mathcal{O} \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded domain, $T > 0$ and $u^i : \mathcal{O} \times [0, T] \times \Omega \to \mathbb{R}$, $i = 1, 2$, be the solutions of the semi-linear initial value problem

$$\begin{cases} du = (a_{11}\Delta u + a_{12}\Delta v + f_1(u, v))dt + \alpha u \circ dW_t \\ dv = (a_{21}\Delta u + a_{22}\Delta v + f_2(u, v))dt + \beta v \circ dW_t \\ u|_{\partial\mathcal{O}} = 0, \quad v|_{\partial\mathcal{O}} = 0 \\ u|_{t=0} = u_0, \ v|_{t=0} = v_0, \end{cases} \tag{4}$$

where $a = (a_{ij})_{1 \le i,j \le 2}$ is a positive definite matrix with real, constant coefficients. Moreover, the constants $\alpha, \beta \in \mathbb{R}$ and the function $f = (f_1, f_2)$ is assumed to be continuously differentiable. We interpret the stochastic system in the sense of Stratonovich and apply an analogous transformation as before. To be more precise, defining the functions $\tilde{u}(t, \omega) := e^{-\alpha W_t(\omega)} u(t, \omega)$ and $\tilde{v}(t, \omega) := e^{-\beta W_t(\omega)} v(t, \omega)$ leads to the following non-autonomous system of random PDEs

$$\begin{cases} \frac{d\tilde{u}}{dt} = a_{11}\Delta\tilde{u} + a_{12}e^{-(\alpha-\beta)W_t}\Delta\tilde{v} + e^{-\alpha W_t} f_1(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v}) \\ \frac{d\tilde{v}}{dt} = a_{21}e^{-(\beta-\alpha)W_t}\Delta\tilde{u} + a_{22}\Delta\tilde{v} + e^{-\beta W_t} f_2(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v}). \end{cases} \tag{5}$$

Random PDEs can be interpreted pathwise and allow to apply deterministic methods. By a generalization of the positivity criterion for semi-linear systems in **[3]** to non-autonomous equations (cf. Part I, Section 2 of our article) we conclude that the solutions of system (5) preserve positivity if and only if the coefficients a_{12} and a_{21} are zero and the interaction terms $F_1(\tilde{u}, \tilde{v}) := e^{-\alpha W_t} f_1(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v})$ and $F_2(\tilde{u}, \tilde{v}) := e^{-\beta W_t} f_2(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v})$ satisfy

$$F_1(0, \tilde{v}) \ge 0, \quad F_2(\tilde{u}, 0) \ge 0$$

for $\tilde{u}, \tilde{v} \geq 0$. Note that due to the particular form of the transformation this is the case if and only if the original reaction functions satisfy

$$f_1(0,v) \geq 0, \quad f_2(u,0) \geq 0$$

for $u, v \geq 0$.

Let us finally discuss the positivity of solutions when the stochastic system (5) is interpreted in the sense of Itô. In this case the system is equivalent to the system of Stratonovich equations

$$\begin{cases} du = (a_{11}\Delta u + a_{12}\Delta v + f_1(u,v) - \frac{\alpha^2}{2}u)dt + \alpha u \circ dW_t \\ dv = (a_{21}\Delta v + a_{22}\Delta u + f_2(u,v) - \frac{\beta^2}{2}v)dt + \beta v \circ dW_t. \end{cases} \tag{6}$$

Analogous transformations as above lead to the following random system for the functions $\tilde{u}$ and $\tilde{v}$

$$\begin{cases} \frac{d\tilde{u}}{dt} = a_{11}\Delta\tilde{u} + a_{12}e^{-(\alpha-\beta)W_t}\Delta\tilde{v} - \frac{\alpha^2}{2}\tilde{u} + e^{-\alpha W_t}f_1(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v}) \\ \frac{d\tilde{v}}{dt} = a_{21}e^{-(\beta-\alpha)W_t}\Delta\tilde{u} + a_{22}\Delta\tilde{v} - \frac{\beta^2}{2}\tilde{v} + e^{-\beta W_t}f_2(e^{\alpha W_t}\tilde{u}, e^{\beta W_t}\tilde{v}). \end{cases} \tag{7}$$

Applying the deterministic positivity criterion we conclude that the positivity of the solutions of system (7) is preserved if and only if the coefficients a_{12} and a_{21} are zero and the interaction functions satisfy

$$\tilde{F}_1(0,\tilde{v}) \geq 0, \quad \tilde{F}_2(\tilde{u},0) \geq 0$$

for $\tilde{u}, \tilde{v} \geq 0$. Here, the modified interaction functions are defined by

$$\tilde{F}_1(\tilde{u},\tilde{v}) := F_1(\tilde{u},\tilde{v}) - \frac{\alpha^2}{2}\tilde{u}, \quad \tilde{F}_2(\tilde{u},\tilde{v}) := F_2(\tilde{u},\tilde{v}) - \frac{\beta^2}{2}\tilde{v}.$$

Hence, due to the linearity of the additional term obtained when using Itô's interpretation this condition is satisfied if and only if the functions F_1 and F_2 fulfil the same property. This in turn is equivalent to an analogous condition for the interaction functions f_1 and f_2 of the unperturbed deterministic system. We summarize our discussion in the following proposition.

Proposition 3. — *The solutions of the system of Stratonovich equations (4) as well as the solutions of the corresponding system obtained through Itô's interpretation of the stochastic system preserve positivity if and only if the solutions of the unperturbed system preserve positivity.*

We want to point out the following: The conditions on the interaction functions f_1 and f_2, which are necessary and sufficient for the positivity of solutions of the unperturbed deterministic system, are equivalent to analogous conditions for the functions F_1 and F_2 appearing in the system of random PDEs in the case of Stratonovich's interpretation. Moreover, they are equivalent to the same conditions on the functions $\tilde{F}_1$ and $\tilde{F}_2$, which we obtain when interpreting the stochastic differential equations in the sense of Itô. Hence, in the particular case of stochastic perturbations by a linear, multiplicative noise the qualitative behaviour of solutions with respect to positivity is not affected - independent of the choice of Itô's or Stratonovich's interpretation.

As was shown above, this is due to the explicit relation between the equations obtained through Itô's, respectively Stratonovich's interpretation, and the particular type of transformation leading to the systems of random PDEs. The necessary and sufficient conditions for the positivity of solutions of the unperturbed system are invariant under all these transformations.

In order to study the general case, where we cannot apply such a simple transformation leading directly to systems of random PDEs, we consider a Wong-Zakaï-type approximation of the stochastic systems of PDEs. As in **[2]** we interpret the stochastic system in the sense of Itô. We essentially use the main result of this article, which states that the solutions of the approximate systems converge in expectation, not to the solution of the original system, but to the solution of a modified one. It turns out that the necessary and sufficient conditions for the positivity as well as for the validity of comparison results are invariant under the transformation relating the original system and the modified system. Moreover, the modified system is exactly the system we obtain when interpreting the original system in the sense of Stratonovich. Hence, we are not only able to derive necessary and sufficient conditions for the positivity and the validity of comparison principles for the solutions of a large class of stochastic PDEs, but also to prove that theses conditions are independent of the choice of Itô's, respectively Stratonovich's, interpretation. That is, the qualitative behaviour of solutions regarding positivity and the validity of comparison principles is independent of the choice of the interpretation for the class of stochastic systems we consider.

3. Main Result

The systems of stochastic PDEs we study are of the following form

$$(8)\ du^l(x,t) = \left(-\sum_{i=1}^{m} A_i^l(x,t,D)u^i(x,t) + f^l(x,t,u(x,t)) \right) dt + \sum_{i=1}^{\infty} q_i g_i^l(x,t,u(x,t)) dW_t^i,$$

where $x \in O, t > 0$ for $l = 1, \ldots, m$. We interpret the stochastic system in the sense of Itô. Here, $u = (u^1, \ldots, u^m)$ is a vector-valued function, $O \subset \mathbb{R}^n$ is a bounded domain and A_i^l are linear elliptic operators of second order. Moreover, we assume $\{W_t^i,\, t \geq 0\}_{i \in \mathbb{N}}$ is a family of independent standard scalar Wiener processes on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ and dW_t^i denotes the corresponding Itô differential. The boundary conditions are given by the operators $(B^1, \ldots, B^m)$,

$$B^l(x,D)u^l(x,t) = 0 \quad \text{on } \partial O,\ t > 0$$

and the solution satisfies the initial conditions

$$u^l(0,x) = u^l_0(x) \quad x \in \bar{O}$$

for $l = 1, \dots, m$.

We denote by (A, f, g) the previous system (8) of Itô equations and the corresponding unperturbed deterministic system by $(A, f, 0)$. In our article we will derive necessary and sufficient conditions for the coefficient functions of the operators A^l_i and the functions f and g to ensure that system (8) preserves the positivity of solutions. In this case, that is, if the solutions corresponding to non-negative initial data remain non-negative as long as they exist, we say that the system satisfies the *positivity property*. The deterministic case has been studied in [**3**]. Assuming that the unperturbed deterministic system $(A, f, 0)$ satisfies the positivity property we are in particular interested in characterizing the class of stochastic perturbations g such that the system (A, f, g) satisfies the positivity property.

In the first section of our article, we prove that necessary and sufficient conditions for the positivity of solutions of the unperturbed system are that the matrices appearing in the differential operator A are diagonal and the components of the interaction function satisfy

$$f^l(x, t, u^1, \dots, u^{l-1}, 0, u^{l+1}, \dots, u^n) \geq 0 \quad \text{for } u^j \geq 0 \text{ and } x \in O, t \geq 0,$$

where $j, l = 1, \dots, m$. This result extends one of the previous results by Efendiev-Sonner (cf. [**3**]). As a consequence, we are led to the study of the following class of systems with diagonal differential operators

$$du^l(x,t) = \Big(-A^l(x,t,D)u^l(x,t) + f^l(x,t,u(x,t))\Big)\, dt + \sum_{i=1}^{\infty} q_i g^l_i(x,t,u(x,t)) dW^i_t, \tag{9}$$

for $l = 1, \dots, m$, where $x \in O$ and $t > 0$. In the sequel we denote by (f, g) the system of SPDEs (9) and the corresponding unperturbed system of PDEs by $(f, 0)$.

As mentioned above the main problem we address in this article is the characterization of stochastic perturbations g such that, if the unperturbed equation satisfies the positivity property, then the perturbed stochastic problem (f, g) satisfies also this property. However, we obtain even a stronger result. We derive necessary and sufficient conditions for the interaction function f and the stochastic perturbation g such that the system of stochastic Itô PDEs (9) satisfies the positivity property. Moreover, the necessary and sufficient conditions for the positivity of solutions, as well as for the validity of comparison theorems, are invariant under the transformation relating the equations obtained through Itô's and Stratonovich's interpretation. As a consequence, our main result is the following:

Theorem 1. — *Let (f, g) be a system of stochastic PDEs in Itô or Stratonovich interpretation. We assume that the functions g^l_j are twice continuously differentiable with respect to u^k,* *for all* $j \in \mathbb{N}$ *and* $k, l = 1, \ldots, m$. *Then, the system* (f, g) *satisfies the positivity property if and only if the interaction function* f *satisfies the deterministic positivity condition and*

$$g_j^l(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m) = 0 \qquad (x, t) \in O \times [0, T],\ u^k \geq 0,$$

for all $j \in \mathbb{N}$ *and* $k, l = 1, \ldots, m$.

The proof makes an essential use of Chueshov-Vuillermot's result in [**2**] on a Wong-Zakaï type approximation theorem for the stochastic problem (f, g). Their main theorem yields a smooth random approximation of the stochastic system, which allows to apply deterministic methods to study the qualitative behaviour of solutions. In particular, we will use a generalization of the necessary and sufficient conditions for the positivity of solutions in the deterministic setting obtained by Efendiev-Sonner in [**3**].

Acknowledgments: We want to thank Tomás Caraballo for helpful discussions.

PART I

THE DETERMINISTIC CASE - NECESSARY AND SUFFICIENT CONDITIONS FOR POSITIVITY

1. Semi-linear Systems of Parabolic PDEs

Our aim is to derive necessary and sufficient conditions for the coefficients of the system of stochastic partial differential equations (8) to satisfy the positivity property. To this end we consider a Wong-Zakaï approximation of this system. The approximation preserves the positivity property of solutions and leads to a family of random equations. To the family of random equations we may apply results from the deterministic theory of PDEs.

Necessary and sufficient conditions for autonomous systems of semi-linear and quasi-linear reaction-diffusion-convection-equations were studied in the article [**3**]. In the present case, we cannot directly apply these results as the Wong-Zakaï approximation leads to a system of random parabolic equations with time-dependent interaction functions. For the convenience of the reader we will present a slight generalization of one of the theorems in [**3**] allowing non-autonomous interactions functions and arbitrary linear elliptic differential operators of second order. The proof uses the same methods and ideas as applied in the mentioned article.

To be more precise, we consider the following class of systems of semi-linear parabolic equations

$$\partial_t u^l(x,t) = -\sum_{i=1}^{m} A_i^l(x,D)u^i(x,t) + f^l(x,t,u(x,t)), \tag{10}$$

where $u = (u^1, \dots, u^m)$ is a vector-valued function of $x \in O$, $t > 0$ and $O \subset \mathbb{R}^n$, $n \in \mathbb{N}$, is a bounded domain with smooth boundary ∂O.

Assumptions on the operator A

The linear second order differential operators $A_i^l(x,D)$ are defined as

$$A_i^l(x,D) = -\sum_{k,j=1}^{n} a_{kj}^{il}(x)\partial_{x_k}\partial_{x_j} + \sum_{k=1}^{n} a_k^{il}(x)\partial_{x_k},$$

for $i, l = 1, \dots, m$.

Comparing with the setting in [**2**] we omit the zero-order linear terms in the operator A as for the problems we address it seems more natural to absorb these terms in the interaction function f.

We assume that the coefficient functions satisfy $a_{kj}^{il} = a_{jk}^{il}$ and the operators are uniformly elliptic, that is

$$\sum_{k,j=1}^{n} a_{kj}^{il}(x)\zeta_k\zeta_j \geq \mu|\zeta|^2, \quad \text{for all } x \in O, \zeta \in \mathbb{R}^n,$$

$i, l = 1 \ldots, m$. Moreover, all coefficient functions are continuously differentiable and bounded in the domain O.

Assumptions on the boundary operators B

The boundary values of the solution are given by the operators

$$B^l(x, D) = b_0^l(x) + \delta^l \sum_{k=1}^{n} b_k^l(x)\partial_{x_k}, \quad l = 1, \ldots, m,$$

where $\delta^l \in \{0, 1\}$. The functions b_k^l, b_0^l are smooth on the boundary ∂O and satisfy $b_0^l \geq 0$. Moreover, we assume $b_0^l \equiv 1$ for $\delta^l = 0$ and $b^l = (b_1^l, \ldots, b_m^l)$ is an outward pointing nowhere tangent vector-field on the boundary ∂O.

Assumptions on the non-linear interaction term f

For the (non-linear) interaction function we assume that the partial derivatives $\partial_u f^l$ exist and are continuous, $l = 1, \ldots, m$. Moreover, we assume that for $x \in O$ and $t > 0$ the functions $f^l = f^l(x, t, u)$ and $\partial_u f^l = \partial_u f^l(x, t, u)$ are bounded for bounded values of u.

2. A Positivity Criterion

In order to formulate our criterion for the positivity of solutions we define the positive cone as the set of componentwise almost everywhere non-negative functions.

Definition 1. — *By $K^+ := \{u : O \to \mathbb{R}^m \mid u^i \in L^2(O),\ u^i \geq 0 \text{ a.e. in } O,\ i = 1, \ldots, m\}$ we denote the* ***positive cone****, that is the set of all non-negative vector-valued functions on the domain O.*

Our concern is not to study the existence of solutions but their qualitative behaviour. Hence, **in the sequel we assume that for any initial data $u_0 \in K^+$ there exists a unique solution and for $t > 0$ the solution satisfies L^∞-estimates**.

The following theorem provides a criterion, which ensures that system (10) satisfies the ***positivity property***, that is, solutions $u(\,\cdot\,, \cdot\,; u_0) : O \times [0, T] \to \mathbb{R}^m$ of system (10), where $T > 0$, originating from non-negative initial data $u_0 \in K^+$ remain non-negative (as long as they exist).

Theorem 2. — *Let the operators A and B be defined as in the beginning of this section and the above conditions on the coefficient functions of the operators and interaction functions be satisfied. Moreover, we assume the initial data $u_0 \in K^+$ is smooth and fulfils the compatibility* *conditions. Then, system (10) satisfies the positivity property if and only if the matrices* $\big(a^{il}_{kj}\big)_{1\le i,l\le m}$ *and* $\big(a^{il}_{k}\big)_{1\le i,l\le m}$ *are diagonal for all* $1 \le j,k \le m$ *and the components of the reaction term satisfy*

$$f^i(x,t,u^1,\dots,\underbrace{0}_{i},\dots,u^m) \ge 0, \quad \text{for } u^1 \ge 0,\dots,u^m \ge 0,$$

$1 \le i \le m$*, and* $x \in O, t > 0$*.*

Hence, concerning stochastic perturbations of these systems of semi-linear PDEs, which we address in the second part of our article, it suffices to study the class of systems (9). A Wong-Zakaï approximation of such systems is considered in the article [**2**], where even slightly more general interaction functions are allowed.

Proof. — Without loss of generality we assume homogeneous Dirichlet boundary values for the solution. For a discussion of general boundary conditions we refer to the article [**3**]. Let us rewrite system (10) in the following form

$$\text{(11)} \qquad \partial_t u(x,t) \;=\; \sum_{k,j=1}^{n} a_{kj}(x)\partial_{x_k}\partial_{x_j}u(x,t) - \sum_{k=1}^{n} a_k(x)\partial_{x_k}u(x,t) + f(x,t,u(x,t)),$$

where the matrices a_{kj} and a_k are defined as

$$a_{kj}(x) \;=\; \begin{pmatrix} a^{11}_{kj}(x) & \cdots & a^{1m}_{kj}(x) \\ \vdots & \ddots & \vdots \\ a^{m1}_{kj}(x) & \cdots & a^{mm}_{kj}(x) \end{pmatrix}, \quad a_k(x) = \begin{pmatrix} a^{11}_{k}(x) & \cdots & a^{1m}_{k}(x) \\ \vdots & \ddots & \vdots \\ a^{m1}_{k}(x) & \cdots & a^{mm}_{k}(x) \end{pmatrix}$$

and all derivatives in system (11) are applied componentwise to the vector-valued function $u = (u^1,\dots,u^l)$.

Necessity: We assume the solution $u(\,\cdot\,,t;u_0)$ corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$ and prove the necessity of the stated conditions. Taking smooth initial data u_0 and an arbitrary function $v \in K^+$, that is orthogonal to u_0 in $L^2(O;\mathbb{R}^m)$, we observe

$$\begin{aligned}\big(\partial_t u|_{t=0}, v\big)_{L^2(O;\mathbb{R}^m)} &= \Big(\lim_{t\to 0_+}\frac{u(\,\cdot\,,t;u_0)-u_0}{t}, v\Big)_{L^2(O;\mathbb{R}^m)} = \\ &= \lim_{t\to 0_+}\Big(\frac{u(\,\cdot\,,t;u_0)}{t}, v\Big)_{L^2(O;\mathbb{R}^m)} - \lim_{t\to 0_+}\Big(\frac{u_0}{t}, v\Big)_{L^2(O;\mathbb{R}^m)} = \\ &= \lim_{t\to 0_+}\Big(\frac{u(\,\cdot\,,t;u_0)}{t}, v\Big)_{L^2(O;\mathbb{R}^m)} \ge 0,\end{aligned}$$

where we used the orthogonality of u_0 and v as well as the assumption $u(\,\cdot\,,t;u_0) \in K^+$. On the other hand, since u is the solution of system (10) corresponding to initial data u_0, we obtain

$$
\begin{aligned}
(12) \qquad \big(\partial_t u|_{t=0}, v\big)_{L^2(O;\mathbb{R}^m)} &= \big(\sum_{k,j=1}^{n} a_{kj}(\cdot)\partial_{x_k}\partial_{x_j} u_0 - \sum_{k=1}^{n} a_k(\cdot)\partial_{x_k} u_0, v\big)_{L^2(O;\mathbb{R}^m)} + \\
&\quad + \big(f(\cdot, 0, u_0), v\big)_{L^2(O;\mathbb{R}^m)} \geq 0.
\end{aligned}
$$

In particular, for fixed $i, l \in \{1, \dots, m\}$, $i \neq l$, choosing the functions $u_0 = (0, \dots, \underbrace{\tilde{u}}_{l}, \dots, 0)$ and $v = (0, \dots, \underbrace{\tilde{v}}_{i}, \dots, 0)$ with $\tilde{u} \geq 0$, $\tilde{v} \geq 0$, leads to the scalar inequality

$$
\int_O \Big(\sum_{k,j=1}^{n} a^{il}_{kj}(x)\partial_{x_k}\partial_{x_j}\tilde{u}(x) - \sum_{k=1}^{n} a^{il}_k(x)\partial_{x_k}\tilde{u}(x) + f^i(x, 0, u_0(x))\Big) \cdot \tilde{v}(x)\, dx \geq 0.
$$

As this inequality holds for an arbitrary non-negative function $\tilde{v} \in L^2(\Omega)$, we obtain the pointwise estimate

$$
(13) \qquad \sum_{k,j=1}^{n} a^{il}_{kj}(x)\partial_{x_k}\partial_{x_j}\tilde{u}(x) - \sum_{k=1}^{n} a^{il}_k(x)\partial_{x_k}\tilde{u}(x) + f^i(x, 0, u_0(x)) \geq 0
$$

almost everywhere in O. This implies for $1 \leq i, l \leq m$, $i \neq l$ and all $1 \leq j, k \leq n$

$$
a^{il}_{kj}(x) = a^{il}_k(x) = 0.
$$

Moreover, $f^i(x, 0, u_0) \geq 0$ for $u_0 = (0, \dots, \underbrace{\tilde{u}}_{l}, \dots, 0)$, $\tilde{u} \geq 0$. Hence, the matrices a_{kj} and a_k are necessarily diagonal. Let us now choose initial data $u_0 = (u^1, \dots, \underbrace{0}_{i}, \dots, u^m)$ and the function $v = (0, \dots, \underbrace{\tilde{v}}_{i}, \dots, 0)$ with $\tilde{u}^1, \dots, \tilde{u}^m \geq 0$, $\tilde{v} \geq 0$ to conclude from inequality (13) that the interaction terms necessarily satisfy

$$
f^i(x, 0, \tilde{u}^1, \dots, 0, \dots \tilde{u}^m) \geq 0,
$$

for $\tilde{u}^1, \dots \tilde{u}^m \geq 0$ and $x \in O$, $1 \leq i \leq m$.

We need to show that these conditions also hold for $t > 0$. Let us suppose that at some time $t_0 > 0$ the solution approaches a boundary point of the positive cone K^+. This implies that $u^i|_{t=t_0} = 0$ for some $1 \leq i \leq m$. Choosing the function $v = (0, \dots, \underbrace{\tilde{v}}_{i}, \dots, 0)$ with arbitrary $\tilde{v} \geq 0$, it is orthogonal to $u(\,\cdot\,, t_0; u_0)$ in $L^2(O;\mathbb{R}^m)$. We observe that

$$
\begin{aligned}
\big(\partial_t u|_{t=t_0}, v\big)_{L^2(O;\mathbb{R}^m)} &= \big(\lim_{t\to(t_0)_+} \frac{u^i(\,\cdot\,, t; u_0) - u^i(\,\cdot\,, t_0; u_0)}{t - t_0}, \tilde{v}\big)_{L^2(O)} = \\
&= \lim_{t\to(t_0)_+} \big(\frac{u^i(\,\cdot\,, t; u_0)}{t - t_0}, \tilde{v}\big)_{L^2(O)} - \lim_{t\to(t_0)_+} \big(\frac{u^i(\,\cdot\,, t_0; u_0)}{t - t_0}, \tilde{v}\big)_{L^2(O)} = \\
&= \lim_{t\to(t_0)_+} \big(\frac{u^i(\,\cdot\,, t; u_0)}{t - t_0}, \tilde{v}\big)_{L^2(O)} \geq 0,
\end{aligned}
$$

where we used that $u^i|_{t=t_0} = 0$ as well as the positivity of the solution, that is $u(\,\cdot\,,t;u_0) \in K^+$ for $t > 0$. On the other hand, as u is a solution of the initial value problem it satisfies

$$(14)\quad \big(\partial_t u|_{t=t_0}, v\big)_{L^2(O;\mathbb{R}^m)} = \Big(\sum_{k,j=1}^{n} a_{kj}(\,\cdot\,)\partial_{x_k}\partial_{x_j} u|_{t=t_0} - \sum_{k=1}^{n} a_k(\,\cdot\,)\partial_{x_k} u|_{t=t_0}, v\Big)_{L^2(O;\mathbb{R}^m)} + \\ + \big(f(\,\cdot\,,t_0,u|_{t=t_0}), v\big)_{L^2(O;\mathbb{R}^m)} \geq 0.$$

By the diagonality of the matrices a_{kj} and a_k and following the same arguments as above we obtain the pointwise inequality

$$f^i(x,t_0,\tilde{u}^1|_{t=t_0},\dots,\underbrace{0}_{i},\dots,\tilde{u}^m|_{t=t_0}) \geq 0$$

almost everywhere in O. Hence, the matrices a_{kj} and a_k are necessarily diagonal and the components of the interaction functions satisfy

$$f^i(x,t,u^1,\dots,\underbrace{0}_{i},\dots,u^m) \geq 0 \qquad \text{for } u^j \geq 0$$

$x \in O, t > 0$ and $1 \leq i,j \leq m$.

Sufficiency: Let us assume the matrices $\big(a^{il}_{kj}\big)_{1\leq i,l\leq m}$ and $\big(a^{il}_{k}\big)_{1\leq i,l\leq m}$ are diagonal and the components of the interaction function satisfy

$$f^l(x,t,u^1,\dots,\underbrace{0}_{i},\dots,u^m) \geq 0, \quad \text{for } u^1 \geq 0,\dots,u^m \geq 0,\ x \in O, t > 0,$$

for all $1 \leq l \leq m$. We show that these conditions ensure that the solution corresponding to initial data $u_0 \in K^+$ remains non-negative for $t > 0$. We will even prove the positivity of solutions under more general assumptions. We allow non-autonomous differential operators and suppose the assumptions on the operator A in the beginning of this section are fulfilled for all $t > 0$. If the differential operators are diagonal, the system of equations takes the form

$$(15)\quad \partial_t u^l(x,t) = \sum_{k,j=1}^{n} a^l_{kj}(x,t)\partial_{x_k}\partial_{x_j} u^l(x,t) - \sum_{k=1}^{n} a^l_k(x,t)\partial_{x_k} u^l(x,t) + f^l(x,t,u(x,t)),$$

$1 \leq l \leq m$, where the functions a^l_{kj} and a^l_k are defined by $a^l_{kj} := a^{ll}_{kj}, a^l_k := a^{ll}_k$. Introducing the positive and negative part $u_+ := \max\{u,0\}$, respectively $u_- := \max\{-u,0\}$, of a given function $u \in L^2(O)$, we can represent it as $u = u_+ - u_-$ and its absolute value as $|u| = u_+ + u_-$. By the definition immediately follows that the product $u_-\,u_+ = 0$. It is a well-known fact that if a function u belongs to the Sobolev space $H^1(O)$, then this also holds for its positive and negative part $u_+, u_- \in \mathrm{H}^1(O)$. Furthermore, the derivatives satisfy

$$Du_- = \begin{cases} -Du & u < 0 \\ 0 & u \geq 0 \end{cases} \qquad Du_+ = \begin{cases} Du & u > 0 \\ 0 & u \leq 0 \end{cases}$$

(cf. [**4**]). This certainly implies

$$Du_+ \, u_- = u_+ \, Du_- = Du_+ \, Du_- = 0.$$

In order to prove the positivity of the solution $u = u(\,\cdot\,,\,\cdot\,; u_0)$ corresponding to initial data $u_0 \in K^+$ we show that $(u_0^i)_- = 0$ a.e. in O implies that $u_-^i := u^i(\,\cdot\,, t; u_0)_- = 0$ a.e. in O for $t > 0$ and all $1 \le i \le m$.

Multiplying the l-th equation of system (15) by u_-^l and integrating over O we obtain

$$\begin{aligned}\big(\partial_t u^l, u_-^l\big)_{L^2(O)} \;&=\; \big(\sum_{k,j=1}^n a_{kj}^l(\cdot,t)\partial_{x_k}\partial_{x_j} u^l, u_-^l\big)_{L^2(O)} - \big(\sum_{k=1}^n a_k^l(\cdot,t)\partial_{x_k} u^l, u_-^l\big)_{L^2(O)} + \\ &\quad + \big(f^l(\cdot,t,u), u_-^l\big)_{L^2(O)},\end{aligned}$$

Note that the left hand side of the equation can be written as

$$\big(\partial_t u^l, u_-^l\big)_{L^2(O)} \;=\; -\big(\partial_t u_-^l, u_-^l\big)_{L^2(O)} = -\frac{1}{2}\partial_t \|u_-^l\|^2_{L^2(O)}.$$

Taking into account the homogeneous Dirichlet boundary conditions we obtain for the first term on the right hand side of the equation

$$\begin{aligned}&\big(\sum_{k,j=1}^n a_{kj}^l(\cdot,t)\partial_{x_k}\partial_{x_j} u^l, u_-^l\big)_{L^2(O)} = -\big(\sum_{k,j=1}^n a_{kj}^l(\cdot,t)\partial_{x_k}\partial_{x_j} u_-^l, u_-^l\big)_{L^2(O)} = \\ &= \int_O \sum_{k,j=1}^n a_{kj}^l(x,t)\partial_{x_j} u_-^l(x,t)\partial_{x_k} u_-^l(x,t)dx + \int_O \sum_{k,j=1}^n \partial_{x_k} a_{kj}^l(x,t)\partial_{x_j} u_-^l(x,t) u_-^l(x,t)dx.\end{aligned}$$

By Young's inequality we derive the estimates

$$\Big|\int_O \sum_{k,j=1}^n \partial_{x_k} a_{kj}^l(x,t)\partial_{x_j} u_-^l(x,t) u_-^l(x,t)dx\Big| \le \epsilon \int_O |\nabla u_-^l(x,t)|^2 dx + C_{\epsilon,1}\|u_-^l\|^2_{L^2(O)},$$

for some constant $C_{\epsilon,1} \ge 0$ and

$$\Big|\int_O \sum_{k=1}^n a_k^l(x,t)\partial_{x_k} u_-^l(x,t) u_-^l(x,t)dx\Big| \le \epsilon \int_O |\nabla u_-^l(x,t)|^2 dx + C_{\epsilon,2}\|u_-^l\|^2_{L^2(O)}dx,$$

for some $C_{\epsilon,2} \ge 0$.

It remains to estimate the interaction term. By assumption the functions f^l are continuously differentiable with respect to u, so we can represent them as

$$f^l(x,t,u^1,\ldots,u^m) = f^l(x,t,u^1,\ldots,\underbrace{0}_{l},\ldots,u^m) + u^l\int_0^1 \partial_{u_l} f^l(x,t,u^1,\ldots,su^l,\ldots,u^m)ds,$$

that is, $f^l(x,t,u^1,\dots,u^m) = f^l(x,t,u^1,\dots,0,\dots,u^m) + u^l \cdot F^l(x,t,u)$, with a bounded function $F^l : \Omega \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$. This representation yields

$$\begin{aligned}
&\int_O f^l(x,t,u(x,t)) \cdot u^l_-(x,t)dx = \int_O f^l(x,t,u^1(x,t),\dots,\underbrace{0}_{l},\dots,u^m(x,t)) \cdot u^l_-(x,t)dx + \\
&+ \int_O u^l(x,t) \cdot F^l(x,t,u(x,t)) \cdot u^l_-(x,t)dx = \\
= \;&\int_O f^l(x,t,u^1(x,t),\dots,\underbrace{0}_{l},\dots,u^m(x,t)) \cdot u^l_-(x,t)dx - \int_O |u^l_-(x,t)|^2 \cdot F^l(x,t,u(x,t))dx.
\end{aligned}$$

Hence, using the uniform parabolicity assumption and collecting all terms we derive the estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t \|u^l_-\|^2_{L^2(O)} + \mu \int_O |\nabla u^l_-(x,t)|^2 dx \le \\
\le \;&\frac{1}{2}\partial_t \|u^l_-\|^2_{L^2(O)} + \int_O \sum_{k,j=1}^n a^l_{kj}(x,t)\partial_{x_j} u^l_-(x,t)\partial_{x_k} u^l_-(x,t)dx \le \\
\le \;&\Big| \int_O \sum_{k,j=1}^n \partial_{x_k} a^l_{kj}(x,t)\partial_{x_j} u^l_-(x,t)u^l_-(x,t)dx + \int_O \sum_{k=1}^n a^l_k(x,t)\partial_{x_k} u^l_-(x,t)u^l_-(x,t)dx \Big| - \\
&- \int_O f^l(x,t,u^1(x,t),\dots,\underbrace{0}_{l},\dots,u^m(x,t)) \cdot u^l_-(x,t)dx + \\
&+ \int_O |u^l_-(x,t)|^2 \cdot F^l(x,t,u(x,t))dx \le \\
\le \;&2\epsilon \int_O |\nabla u^l_-(x,t)|^2 dx + (C_\epsilon + C)\|u^l_-\|^2_{L^2(O)} - \\
&- \int_O \big(f^l(x,t,u^1(x,t),\dots,0,\dots,u^m(x,t))\big)u^l_-(x,t)dx,
\end{aligned}$$

for some constants $C_\epsilon, C \ge 0$. Under the assumption $u^j \ge 0, j \ne l$, the conditions imposed on the interaction functions imply

$$f^l(x,t,u^1(x,t),\dots,\underbrace{0}_{l},\dots,u^m(x,t)) \cdot u^l_-(x,t) \ge 0.$$

Choosing $\epsilon > 0$ sufficiently small we therefore obtain the inequality

$$\partial_t \|u^l_-\|^2_{L^2(O)} \le c \cdot \|u^l_-\|^2_{L^2(O)},$$

for some constant $c \ge 0$. By Gronwall's Lemma and the initial condition $(u^l_0)_- = 0$ we conclude that $u^l_- = 0$ a.e. in O for $t > 0$.

It remains to justify our assumptions on the interaction functions. Instead of the original system (11) we consider the modified system

$$\begin{cases} \partial_t \hat{u}(x,t) = \sum_{k,j=1}^{n} a_{kj}(x,t)\partial_{x_k}\partial_{x_j}\hat{u}(x,t) - \sum_{k=1}^{n} a_k(x,t)\partial_{x_k}\hat{u}(x,t) + \hat{f}(x,t,\hat{u}(x,t)) \\ \hat{u}|_{t=0} = u_0 \\ \hat{u}|_{\partial O} = 0, \end{cases}$$

where the function $\hat{f}$ is given by

$$\hat{f}^l(x,t,\hat{u}(x,t)) = f^l(x,t,|\hat{u}^1(x,t)|,\ldots,0,\ldots,|\hat{u}^m(x,t)|) + \hat{u}^l(x,t)\cdot F^l(x,t,\hat{u}(x,t))$$

with F^l as defined above. Following the same arguments as before we conclude that the solution $\hat{u}$ of this modified system preserves positivity, that is, if the initial data $u_0 \in K^+$ we obtain $\hat{u}(\,\cdot\,,t;u_0) \in K^+$ for $t > 0$. However, the solution $\hat{u}$ with $\hat{u}^1 \geq 0, \ldots, \hat{u}^m \geq 0$ satisfies the original system

$$\begin{cases} \partial_t u(x,t) = \sum_{k,j=1}^{n} a_{kj}(x,t)\partial_{x_k}\partial_{x_j}u(x,t) - \sum_{k=1}^{n} a_k(x,t)\partial_{x_k}u(x,t) + f(x,t,u(x,t)) \\ u|_{t=0} = u_0 \\ u|_{\partial O} = 0. \end{cases}$$

By the uniqueness of the solution corresponding to initial data u_0 follows $u = \hat{u}$, which implies that the solution u of the original system satisfies $u(\,\cdot\,,t;u_0) \in K^+$ for $t > 0$ and concludes the proof of the theorem. □

Remark 1. — *Under appropriate assumptions on the time-dependent coefficients of the operator A we could prove an analogous result for non-autonomous differential operators. As the proof of the sufficiency of the stated conditions does not change in the non-autonomous setting we decided to included this generalization in the proof of Theorem 2. The first part of the proof however requires some modifications and additional hypothesis. As these questions are not the main concern of our article we decided to state the theorem in the above form for autonomous differential operators.*

PART II

THE STOCHASTIC CASE - NECESSARY AND SUFFICIENT CONDITIONS FOR POSITIVITY AND VALIDITY OF COMPARISON PRINCIPLES

In this section we derive necessary and sufficient conditions for a large class of stochastic systems of PDEs to satisfy the positivity property. The main ingredient of our proof is a Wong-Zakaï-type approximation theorem, which allows us to associate to a given stochastic system of PDEs a family of random PDEs. The solutions of the family of random equations converge in expectation to the solution of the original problem. As a consequence, we can apply the result for deterministic systems stated in the first part of our article to obtain a criterion for the positivity of solutions of stochastic systems.

1. Wong-Zakaï Approximation and Random Systems of PDEs

In 1965 E. Wong and M. Zakaï ([**9**],[**10**]) studied the relation between ordinary and stochastic differential equations. The main point is that Itô's approach to stochastic differential equations is based on Itô's definition of stochastic integrals, which are not directly connected to the limit of ordinary integrals. In particular, stochastic differential equations are not defined as an extension or limit of ordinary differential equations. Wong and Zakaï introduce a smooth approximation of the Brownian motion in order to obtain an approximation of stochastic integrals by ordinary integrals. Doing so, they obtain an approximation of the stochastic differential equation by a family of random differential equations. However, when the smoothing parameter tends to zero the random solutions do not converge to a solution of the original stochastic differential equation, but a modified one. The appearing correction term is called *Wong-Zakaï correction term.* The Wong-Zakaï approximation theorem has been generalized in various directions. We refer to D.W. Stroock and S.R.S. Varadhan [**7**] for systems of ordinary differential equations and to G. Tessitore and J. Zabczyk [**8**] for evolution equations in an abstract setting. In this section, we briefly recall the main result by Chueshov-Vuillermot in [**2**] about a Wong-Zakaï-type approximation theorem for a class of stochastic systems of semi-linear parabolic PDEs.

To be more precise, we will analyse the class of systems of stochastic Itô PDEs (9), where the operators in the deterministic part of the equation are defined as in Part I, Section 1.

Assumptions on stochastic perturbations

We assume $\{W_t^j,\, t \geq 0\}_{j\in\mathbb{N}}$ is a family of mutually independent standard scalar Wiener processes on the canonical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ and dW_t^j denotes the corresponding Itô differential. The non-negative parameters q_j are normalization factors. Moreover, the functions $g_j^l : O \times [0,T] \times \mathbb{R} \to \mathbb{R}$ are smooth and assumed to be bounded for bounded values of the solution, $j \in \mathbb{N}, l = 1, \ldots, m$.

1.1. Smooth Predictable Approximation of the Wiener Process. — A general notion of a smooth predictable approximation of the Wiener process is defined by Chueshov and Vuillermot in **[2]** (Definition 4.1, p.1440). In the following, we will take the main example provided in this article as a definition (Proposition 4.2, p.1441).

Let $\{W_t, t \geq 0\}$ be a standard scalar Wiener process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t; t \in \mathbb{R}_+\}$. The ***smooth predictable approximation*** of $\{W_t, t \geq 0\}$ is the family of random processes $\{W_\epsilon(t), t \geq 0\}_{\epsilon > 0}$ defined by

$$W_\epsilon(t) = \int_0^\infty \phi_\epsilon(t - \tau) W_\tau d\tau,$$

where $\phi_\epsilon(t) = \epsilon^{-1}\phi(t/\epsilon)$ and $\phi(t)$ is a function with the properties

$$\phi \in C^1(\mathbb{R}), \ \text{supp}\phi \subset [0,1], \ \int_0^1 \phi(t) dt = 1.$$

We will need the following result (**[2]**, p.1442), which states that the derivative of the smooth predictable approximation W_ϵ, denoted by $\dot{W}_\epsilon$, can be written as a stochastic integral of the form

$$\dot{W}_\epsilon(t) = \int_{t-\epsilon}^t \phi_\epsilon(t - \tau) dW_\tau, \ t \geq \epsilon.$$

As a consequence, $\dot{W}_\epsilon$ is Gaussian, which will be fundamental in our proof.

1.2. Predictable Smoothing of Itô's Problem and Random Systems. — Using the previously defined family of smooth predictable approximations $\{W_\epsilon^j(t), t \geq 0\}_{\epsilon > 0, j \in \mathbb{N}}$ of the Wiener processes $\{W_t^j, t \geq 0\}_{j \in \mathbb{N}}$ allows us to define the predictable smoothing of Itô's problem (9) as the family of random equations

$$(16) \quad du^l(x,t) = \big(-A^l(x,t,D)u^l(x,t) + f^l(x,t,u(x,t))\big)dt + \big(\sum_{j=1}^\infty q_j g_j(x,t,u(x,t)) \dot{W}_\epsilon^j(t)\big)dt,$$

where $l = 1, \ldots, m$. As a consequence, using our notation, we are led to the following definition:

Definition 2. — *The* ***smooth approximation of the stochastic system*** (f, g) *of PDEs with respect to the smooth predictable approximation* $\{W_\epsilon(t), t \geq 0\}_{\epsilon > 0}$ *is defined as the family* *of random PDEs* $(f_{\epsilon,\omega},0)$*, where*

$$f^l_{\epsilon,\omega}(x,t,u(x,t)) = f^l(x,t,u(x,t)) + \sum_{j=1}^{\infty} q_j g^l_j(x,t,u(x,t))\dot{W}^j_\epsilon(t),$$

$\omega\in\Omega,\ \epsilon>0.$

1.3. A Wong-Zakaï Approximation Theorem. — Following Chueshov and Vuillermot ([**2**], p.1436) we use the following notion of *mild solution* for a stochastic system of PDEs (f,g):

Definition 3. — *A random function* $u(x,t,\omega) = (u^1(x,t,\omega),\dots,u^m(x,t,\omega))$ *is said to be a* ***mild solution*** *of* (f,g) *in the space* $V = W^1_{2,B}(O,\mathbb{R}^m)$ *on the interval* $[0,T]$*, if* $u(t) = u(x,t,\omega)\in C(0,T;L^2(\Omega\times O))$ *is a predictable process such that*

$$\int_0^T E \parallel u(t) \parallel_V^2 dt < \infty$$

and satisfies the integral equation

$$u(t) = U(t,0)u_0 + \int_0^t U(t,\tau)f(\tau,u(\tau))d\tau + \sum_{j=1}^{\infty} q_j \int_0^t U(t,\tau)g_j(\tau,u(\tau))dW^j(\tau,\omega), \tag{17}$$

where we assume that all integrals in (17) exist.

The family $\{U(t,\tau),\ 0\le\tau\le t<\infty\}$ in the above definition denotes the linear evolution family generated by the operators $\{A(t), t\ge 0\}$ in $L^2(O;\mathbb{R}^m)$. The domain of the linear operators is defined as

$$W^2_{2,B}(O;\mathbb{R}^m) := \{u\in W^{2,2}(O;\mathbb{R}^m)\ :\ Bu=0\},$$

where B denotes the boundary operator and

$$W^{2,2}(O) := \{u\in L^2(O)\ :\ D^\alpha u\in L^2(O) \text{ for all } |\alpha|\le k\}.$$

For further details we refer to [**2**] and [**1**]. Next, we introduce the notion of convergence that we will use.

Definition 4. — *Let* (f,g) *be a stochastic system of PDEs and* $(f_{\epsilon,\omega},0)$ *be its smooth approximation. We say that a mild solution* u_ϵ *of the random system* $(f_{\epsilon,\omega},0)$ *converges to a mild solution* $\hat{u}$ *of the stochastic system of PDEs* $(\hat{f},g)$ *if*

$$\lim_{\epsilon\to 0}\int_0^T E \parallel \hat{u}(t) - u_\epsilon(t) \parallel^2_{W^1_2(O;\mathbb{R}^m)} dt = 0.$$

The main result of Chueshov and Vuillermot in [**2**] is the following (Theorem 4.3, p.1443):

Theorem 3. — *Assume that the stated assumptions on the operators A and B and the functions f and g are satisfied. Moreover, let $\sum_{j=1}^{\infty} q_j < \infty$, the initial data $u_0 \in C_B^2(O;\mathbb{R}^m)$ be $\mathcal{F}_0$-measurable and $E\|u_0\|_{C^2(O)}^r < \infty$ for some $r > 8$. We assume the associated system of random PDEs $(f_{\epsilon,\omega}, 0)$ has a mild solution u_ϵ belonging to the class $C(0,T;L^r(\Omega, X_{\alpha,p}))$ for all $0 \le \alpha < 1$ and $p > 1$ and for this solution there exists a constant C independent of ϵ such that*

$$\sup_{t\in[0,T]} E \parallel u_\epsilon \parallel_{L^p(O)}^r \le C \text{ for all } p > 1.$$

Then, the mild solution u_ϵ converges to a solution u_{cor} of the corrected stochastic system of PDEs (f_{cor}, g) when ϵ tends to zero, where

$$f_{\mathrm{cor}}^l = f^l + \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g_j^i \frac{\partial g_j^l}{\partial u^i},$$

for $l = 1, \ldots m$.

For further details we refer to the article [**2**]. Our aim is not to prove the existence of solutions, we are interested in their qualitative behaviour. Hence, in the sequel we assume that a solution of the stochastic initial value problem exists and the solution of the modified system is given as the limit of the solutions of the smooth random approximations. Sufficient conditions for existence and uniqueness of solutions can be found in the cited article.

2. How to Study the Qualitative Behaviour of Solutions of Systems of Stochastic PDEs

We will now apply the Wong-Zakaï approximation theorem and the results for the positivity of solutions of deterministic systems in the first part of our article to study the qualitative behaviour of solutions of systems of stochastic PDEs (f, g).

2.1. The General Strategy. — The general strategy in all proofs is the following:

- As Chueshov and Vuillermot in [**2**] we associate to a system (F, g) of stochastic PDEs a system of random PDEs. The system of random PDEs is explicit and depends on the definition of the smooth approximation W_ϵ of the Wiener process $\{W(t), t \ge 0\}$. It is given by the family of random PDEs $(F_{\epsilon,\omega}, 0)$, $\epsilon > 0$, $\omega \in \Omega$, where

$$F_{\epsilon,\omega}^l = F^l + \sum_{j=1}^{\infty} q_j g_j^l \dot{W}_\epsilon^j,$$

  for $l = 1, \ldots, m$, and $\dot{W}_\epsilon$ denotes the time derivative of the smooth predictable approximation W_ϵ. The Wong-Zakaï approximation theorem states that the solutions of the random system of PDEs converge in expectation to the solutions of the modified system of stochastic PDEs

$$\big(F + \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 h_j, g\big),$$

where the function $h_j = (h_j^1, \ldots, h_j^m)$ is given by

$$h_j^l(x,t,u(x,t)) = \sum_{i=1}^{m} g_j^i(x,t,u(x,t)) \frac{\partial g_j^l}{\partial u^i}(x,t,u(x,t)).$$

– Consequently, for a given stochastic system (f,g) we first construct a system of stochastic PDEs (F,g) such that the solutions of its associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ converge to the solutions of our original system (f,g) of stochastic PDEs.
– We then use results from the deterministic theory for the qualitative behaviour of solutions of the system of random PDEs $(F_{\epsilon,\omega}, 0)$ and prove that this property is preserved by passing to the limit when ϵ goes to zero.

2.2. An Example: A Comparison Principle and Sufficient Conditions by Chueshov-Vuillermot. — In this section we present sufficient conditions for the function g to ensure that, if the solutions of the unperturbed system $(f,0)$ satisfy a comparison principle, then this property is preserved by the solutions of the perturbed stochastic system.

For two vectors $u, v \in \mathbb{R}^m$ we write $u \le v$ if this order relation holds componentwise, that is $u^i \le v^i$ for all $i = 1, \ldots, m$. In order to formulate the result we introduce the following notions.

Definition 5. — *We say that the deterministic system $(f,0)$ satisfies the **comparison principle**, if for given initial data such that $u_0(x) \le v_0(x)$ holds a.e. in O the corresponding solutions $u = (\cdot,\cdot;u_0)$ and $v = v(\cdot,\cdot;v_0)$ preserve this order relation, that is $u^i(x,t) \le v^i(x,t)$ holds a.e. in $O \times [0,T]$, for all $i = 1, \ldots, m$. In an analogous manner the notion of the comparison principle is defined for stochastic systems (f,g).*

*Furthermore, we call a function $f : O \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^m$ **quasi-monotone**, if it satisfies*

$$f^l(x,t,u) \le f^l(x,t,v)$$

for all $= 1, \ldots, m$, $(x,t) \in O \times [0,T]$ and all $u, v \in \mathbb{R}^m$ such that $u \le v$ and $u^l = v^l$.

The following result is by Chueshov-Vuillermot ([**2**], Theorem 5.8, p.1479), although we formulate it in a different way in order to adapt it to the notions of our article.

Theorem 4. — *Let (f,g) be a system of stochastic PDEs such that the function f is quasi-monotone. We assume that each of the functions g_j^l depends on the component u^l of the solution only, that is $g_j^l(x,t,u) = g_j^l(x,t,u^l)$ for $l = 1, \ldots, m$, $j \in \mathbb{N}$. Then, the stochastic* *system (f, g) satisfies the comparison principle.*

It is a well-known result in the deterministic theory of PDEs that the quasi-monotonicity of the interaction function f ensures that the system $(f, 0)$ satisfies the comparison principle. Hence, the condition on the functions g_j^l stated in the previous theorem is sufficient to ensure the persistence of this property of the solutions under stochastic perturbations. As mentioned before, this theorem was stated in [**2**]. However, we present a slightly modified proof in order to illustrate the general strategy outlined in the previous section.

Proof. — **Step 1 - Quasi-monotonicity of the associated unperturbed system of stochastic PDEs**

We are looking for a function F such that $F + \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 h_j = f$, where the functions h_j were defined in the previous section. This implies

$$F^l = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 h_j^l = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \sum_{i=1}^{m} g_j^i \frac{\partial g_j^l}{\partial u^i} = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 g_j^l \frac{\partial g_j^l}{\partial u^l},$$

$l = 1, \dots, m$, $j \in \mathbb{N}$, because the functions g_j^l depend on the component u^l of the solution only.

Furthermore, f is assumed to be quasi-monotone, that is

$$f^l(x, t, v^1, \dots, v^{l-1}, \tilde{u}, v^{l+1}, \dots, v^m) \le f^l(x, t, u^1, \dots, u^{l-1}, \tilde{u}, u^{l+1}, \dots, u^m),$$

for all $(x, t) \in O \times [0, T]$, whenever $v^k \le u^k$ for $k = 1, \dots, m$. Due to the assumption that the functions g_j^l are independent of u^k, $k \ne l$, it therefore follows

$$\begin{aligned} & f^l(x, t, v^1, \dots, v^{l-1}, \tilde{u}, v^{l+1}, \dots, v^m) - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 g_j^l(x, t, \tilde{u}) \frac{\partial g_j^l}{\partial \tilde{u}}(x, t, \tilde{u}) \le \\ \le \quad & f^l(x, t, u^1, \dots, u^{l-1}, \tilde{u}, u^{l+1}, \dots, u^m) - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 g_j^l(x, t, \tilde{u}) \frac{\partial g_j^l}{\partial \tilde{u}}(x, t, \tilde{u}), \end{aligned}$$

that is, the function F is quasi-monotone as well,

$$F^l(x, t, v^1, \dots, v^{l-1}, \tilde{u}, v^{l+1}, \dots, v^m) \le F^l(x, t, u^1, \dots, u^{l-1}, \tilde{u}, u^{l+1}, \dots, u^m),$$

for $v^k \le u^k$ and all $(x, t) \in O \times [0, T]$, $l = 1, \dots m$.

Step 2 - Preservation of quasi-monotonicity by the system of random PDEs

The associated system of random PDEs for (F, g) is given by $(F_{\epsilon,\omega}, 0)$, where the function $F_{\epsilon,\omega}^l = F^l + \sum_{j=1}^{\infty} q_j g_j^l \dot{W}_\epsilon^j$. As the smooth approximations W_ϵ^j of the Wiener process depend only on the time t and $\omega \in \Omega$, and the functions g_j^l depend only on the component u^l of the solution, for $l = 1, \dots, m$, $j \in \mathbb{N}$, it follows

$$F^l(x,t,v^1,\dots,v^{l-1},\tilde{u},v^{l+1},\dots,v^m) + \sum_{j=1}^{\infty} q_j g_j^l(x,t,\tilde{u})\dot{W}_\epsilon^j(t) \leq$$
$$\leq \; F^l(x,t,u^1,\dots,u^{l-1},\tilde{u},u^{l+1},\dots,u^m) + \sum_{j=1}^{\infty} q_j g_j^l(x,t,\tilde{u})\dot{W}_\epsilon^j(t),$$

for all $(x,t) \in O \times [0,T]$ and $v^k \leq u^k$, where we used the quasi-monotonicity of the function F. This is equivalent to

$$F^l_{\epsilon,\omega}(x,t,v^1,\dots,v^{l-1},\tilde{u},v^{l+1},\dots,v^m) \leq F^l_{\epsilon,\omega}(x,t,u^1,\dots,u^{l-1},\tilde{u},u^{l+1},\dots,u^m),$$

for all $\epsilon > 0$ and $\omega \in \Omega$. Hence, the property of quasi-monotonicity is preserved by the system of random PDEs.

To the system of random PDEs $(F_{\epsilon,\omega}, 0)$ we may apply the comparison theorem for deterministic systems. As a consequence, if the initial data u_0 and v_0 satisfy $u_0(x,\omega) \leq v_0(x,\omega)$ for almost all $x \in O$, $\omega \in \Omega$, we conclude

$$u_\epsilon(x,t,\omega) \leq v_\epsilon(x,t,\omega)$$

for all $\epsilon > 0$, $\omega \in \Omega$ and almost all $(x,t) \in O \times [0,T]$.

Step 3 - Passage to the limit when ϵ goes to zero

By the Wong-Zakaï approximation theorem and our construction of the function F, the solutions of the system of random PDEs $(F_{\epsilon,\omega}, 0)$, $\omega \in \Omega$, converge in expectation to the solution of the initial system of stochastic PDEs (f,g). Hence, taking the limit when ϵ goes to zero, we obtain

$$u(t,x) \leq v(t,x), \; (t,x) \in O \times [0,T],$$

almost surely, which concludes the proof of the theorem. □

3. A Positivity Criterion for Systems of Stochastic PDEs

In order to study the positivity of solutions of a given stochastic system (f,g) of PDEs we follow the strategy outlined in Part II, Section 2. For a general stochastic perturbation determined by the functions $g_j = (g_j^1, \dots, g_j^m)$, $j \in \mathbb{N}$, the associated unperturbed system $(F,0)$ of PDEs is given by

$$F^l = f^l - \frac{1}{2}\sum_{j=1}^{\infty} q_j^2 \left(g_j^1 \frac{\partial g_j^l}{\partial u^1} + \dots + g_j^m \frac{\partial g_j^l}{u^m} \right).$$

In the first part of our article we proved that the deterministic system $(f,0)$ satisfies the positivity property if and only if the components f^l of the interaction function satisfy

(18) $\quad f^l(x,t,u^1,\dots,u^{l-1},0,u^{l+1},\dots,u^m) \geq 0, \;$ for all $(x,t) \in O \times [0,T], \; u^k \geq 0,$

for $k, l = 1, \ldots, m$. This motivates the following definition.

Definition 6. — *We say that the function*

$$f : O \times [0,T] \times \mathbb{R}^m \to \mathbb{R}^m,\ f(x,t,u) = (f^1(x,t,u), \ldots, f^m(x,t,u)),$$

satisfies the ***positivity condition*** *if all components f^l, $1 \le l \le m$ satisfy the property (18).*

The following lemma will be essential for the proof of our main result.

Lemma 1. — *Let (f, g) be a given stochastic system of PDEs. We assume that the functions g^l_j are twice continuously differentiable with respect to u^k and satisfy*

$$g^l_j(x,t,u^1,\ldots,u^{l-1},0,u^{l+1},\ldots,u^m) = 0$$

for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$. Then, the following statements are equivalent:

(a) *The function f satisfies the positivity condition.*

(b) *The modified function F satisfies the positivity condition.*

(c) *The associated random functions $F_{\epsilon,\omega}$ satisfy the positivity condition for all $\epsilon > 0$ and $\omega \in \Omega$.*

Proof. — The proof is a simple computation. As the functions g^l_j are continuously differentiable with respect to u^l and $g^l_j(x,t,u^1,\ldots,u^{l-1},0,u^{l+1},\ldots,u^m) = 0$ we can represent them in the form $g^l_j(x,t,u) = u^l G^l_j(x,t,u)$ with a continuously differentiable function G^l_j, for all $j \in \mathbb{N}$ and $l = 1, \ldots, m$. Consequently, we obtain for the sum appearing in the Wong-Zakaï correction term

$$\sum_{i=1}^m g^i_j \frac{\partial g^l_j}{\partial u^i} = \sum_{i=1}^m g^i_j \frac{\partial (u^l G^l_j)}{\partial u^i} = \sum_{i \ne l} g^i_j u^l \frac{\partial G^l_j}{\partial u^i} + g^l_j \frac{\partial (u^l G^l_j)}{\partial u^l},$$

which leads to an associated function F of the form

$$F^l = f^l - \frac{1}{2} \sum_{j=1}^\infty q_j^2 \sum_{i=1}^m g^i_j \frac{\partial g^l_j}{\partial u^i} = f^l - \frac{1}{2} \sum_{j=1}^\infty q_j^2 \left(\sum_{i \ne l} g^i_j u^l \frac{\partial G^l_j}{\partial u^i} + g^l_j \frac{\partial (u^l G^l_j)}{\partial u^l} \right).$$

Due to the assumption $g^l_j(x,t,u^1,\ldots,u^{l-1},0,u^{l+1},\ldots,u^m) = 0$ we note that the function F satisfies the positivity condition if and only if f satisfies the positivity condition as all correction terms vanish when $u^l = 0$. Finally, the associated system of random PDEs $(F_{\epsilon,\omega}, 0)$ is given by

$$F^l_{\epsilon,\omega} = F^l + \sum_{j=1}^\infty q_j g^l_j \dot{W}^j_\epsilon.$$

The imposed condition on the functions g^l_j therefore implies

$$F^l_{\epsilon,\omega}(x,t,v) = F^l(x,t,v) = f^l(x,t,v),$$

where $v := (x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m)$, $1 \le l \le m$, which proves that f satisfies the positivity condition if and only if $F_{\epsilon,\omega}$ satisfies the positivity condition and concludes the proof of the lemma. □

Applying Lemma 1 we are in position to prove our main result. The following theorem yields necessary and sufficient conditions for the stochastic system (f, g) to satisfy the positivity property. In particular, it allows us to characterize the class of stochastic perturbations such that the solutions of (f, g) preserve positivity.

Theorem 5. — *Let (f, g) be a system of stochastic PDEs. We assume that the functions g_j^l are twice continuously differentiable with respect to u^k, for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$. Then, the system (f, g) satisfies the positivity property if and only if the interaction function f satisfies the positivity condition and*

$$g_j^l(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m) = 0 \qquad (x, t) \in O \times [0, T],\ u^k \ge 0,$$

for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$.

Proof. — ***Sufficiency:*** We assume system (f, g) satisfies the stated conditions. In order to show the positivity of solutions we follow the same strategy as in the proof of Theorem 4. First, associate to the given system (f, g) the unperturbed system of PDEs $(F, 0)$ and consider the corresponding family of random approximations $(F_{\epsilon,\omega}, 0)$. The function g satisfies the hypothesis of Lemma 1 and f the positivity condition, so it follows that the random functions $F_{\epsilon,\omega}$ satisfy the positivity condition for all $\epsilon > 0$ and $\omega \in \Omega$. The solutions of the random system $(F_{\epsilon,\omega}, 0)$ converge in expectation to the solution of the original system (f, g) by Theorem 3. Hence, the positivity of the solutions follows exactly as in the proof of Theorem 4.
Necessity: We assume the solution of the stochastic system (f, g) remains non-negative for $t > 0$. In order to show that the stated conditions are necessary we first prove that if

$$g_j^l(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m) \ne 0,$$

the system $(F_{\epsilon,\omega}, 0)$ does not satisfy the positivity property. In fact, if the random system $(F_{\epsilon,\omega}, 0)$ satisfies the positivity property, the function $F_{\epsilon,\omega}$ necessarily satisfies the positivity condition. Consequently, for all $l = 1, \ldots, m$ and $(x, t) \in O \times [0, T]$ the inequality

$$F_{\epsilon,\omega}^l(x, t, v) = F^l(x, t, v) + \sum_{j=1}^{\infty} q_j g_j^l(x, t, v) \dot{W}_\epsilon^j(t) \ge 0 \tag{19}$$

holds, where $v := (u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m)$ with $u^1, \ldots, u^m \ge 0$. The derivative of the smooth approximation $W_\epsilon(t)$ of the Wiener process takes arbitrary values. Hence, if we assume that $g_j^l(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m)$ is different from zero, then for sufficiently small $\epsilon > 0$ we always find an $\omega \in \Omega$ such that inequality (19) is violated. Consequently, the random system $(F_{\epsilon,\omega}, 0)$ does not satisfy the positivity property, which contradicts our assumption and proves that $g_j^l(x, t, u^1, \ldots, u^{l-1}, 0, u^{l+1}, \ldots, u^m) = 0$ for all $j \in \mathbb{N}, 1 \leq l \leq m$. Furthermore, if the functions g_j^l satisfy this condition, inequality (19) now implies that the function F necessarily satisfies the positivity condition. By Lemma 1, this is the case if and only if the interaction function f of the original system satisfies the positivity condition, which concludes the proof of our theorem. □

Next, we prove that the same result (see below Corollary 1) is valid if we apply Stratonovich's interpretation of stochastic differential equations. In other words, the positivity property of solutions is independent of the choice of interpretation, which is the statement of Theorem 1 in the introduction.

Corollary 1. — *Let (f, g) be a system of stochastic (Itô) PDEs. We assume that the functions g_j^l are twice continuously differentiable with respect to u^k, for all $j \in \mathbb{N}$ and $k, l = 1, \ldots, m$. Then, the corresponding system obtained when using Stratonovich's interpretation of the stochastic system satisfies the positivity property if and only if the functions f and g satisfy the conditions of the previous theorem.*

Proof. — The Wong-Zakaï correction term coincides with the transformation relating Ito's and Stratonovich's interpretation of the stochastic system. That is, the solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ converge to the solution of the given stochastic system, when interpreted in the sense of Stratonovich. Hence, the statement of the corollary is an immediate consequence of Theorem 5 and Lemma 1. □

4. Necessary and Sufficient Conditions for Comparison Principles

As a direct consequence of the positivity criterion we obtain necessary and sufficient conditions for the stochastic system to satisfy the comparison principle.

Theorem 6. — *Let (f, g) be a system of stochastic PDEs. We assume that the functions g_j^l are twice continuously differentiable with respect to u^k, for all $j \in \mathbb{N}$ and $1 \leq k, l \leq m$. Then, the system (f, g) satisfies the comparison principle if and only if the interaction function f is quasi-monotone and the functions g_j^l depend on the component u^l of the solution only, that is*

$$g_j^l(x, t, u^1, \ldots, u^m) = g_j^l(x, t, u^l)$$

for all $j \in \mathbb{N}$, $1 \leq l \leq m$.

Proof. — Let u_0 and v_0 be given initial data satisfying $u_0(x, \omega) \geq v_0(x, \omega)$ for all $x \in O, \omega \in \Omega$. Applying Theorem 5 we derive necessary and sufficient conditions to ensure that the order is preserved by the corresponding solutions. As the differential operator A is linear, the function $w := u - v$ is a solution of the stochastic system $(\tilde{f}, \tilde{g})$ with

$$\tilde{f}^l(x,t,w) := f^l(x,t,u) - f^l(x,t,v) \quad \text{and} \quad \tilde{g}^l_j(x,t,w) := g^l_j(x,t,u) - g^l_j(x,t,v)$$

for $j \in \mathbb{N}$, $1 \leq l \leq m$. Furthermore, by the definition of w the original system (f,g) satisfies the comparison principle if and only if the system $(\tilde{f}, \tilde{g})$ satisfies the positivity property. Theorem 5 yields necessary and sufficient conditions for the latter. Namely, system $(\tilde{f}, \tilde{g})$ satisfies the comparison principle if and only if the function $\tilde{f}$ satisfies the positivity condition and

$$\tilde{g}^l_j(x,t,w^1,\ldots,w^{l-1},0,w^{l+1},\ldots,w^m) = 0$$

holds for all $j \in \mathbb{N}$ and $1 \leq l \leq m$. The positivity condition for $\tilde{f}$ is equivalent to the quasi-monotonicity of the function f, which proves the stated condition on the interaction function. Moreover, the function $\tilde{g}$ satisfies

$$\tilde{g}^l_j(x,t,w^1,\ldots,w^{l-1},0,w^{l+1},\ldots,w^m) = 0$$

for $w \geq 0$ if and only if the equality

$$\tilde{g}^l_j(x,t,u^1,\ldots,u^{l-1},\tilde{u},u^{l+1},\ldots,u^m) = g^l_j(x,t,v^1,\ldots,v^{l-1},\tilde{u},v^{l+1},\ldots,v^m)$$

holds for all $\tilde{u} \in \mathbb{R}$, $u \geq v$. This shows that the functions g^l_j depend on the component u^l of the solution only. □

Theorem 6 shows that the conditions imposed on the stochastic perturbation g and the interaction function f by Chueshov and Vuillermot in Theorem 4 are not only sufficient but also necessary to ensure that the stochastic system satisfies the comparison principle. As in the case of positivity, the necessary and sufficient conditions for the comparison principle are also valid, when the stochastic system is interpreted in the sense of Stratonovich.

Corollary 2. — *Let (f,g) be a system of stochastic PDEs. We assume that the functions g^l_j are twice continuously differentiable with respect to u^k, for all $j \in \mathbb{N}$ and $1 \leq k,l \leq m$. Then, the corresponding system obtained when using Stratonovich's interpretation of the stochastic system satisfies the comparison principle if and only if the functions f and g satisfy the conditions of the previous theorem.*

Proof. — From the proof of Theorem 4 we deduce that, if the stochastic perturbations g^l_j, $j \in \mathbb{N}, l = 1,\ldots,m$, depend on the component u^l of the solution only, then the following statements are equivalent:

(a) The function f is quasi-monotone.
(b) The modified function F is quasi-monotone.
(c) The associated random functions $F_{\epsilon,\omega}$ are quasi-monotone for all $\epsilon > 0$ and $\omega \in \Omega$.

The solutions of the random approximations $(f_{\epsilon,\omega}, 0)$ converge to the solution of the given stochastic system, when interpreted in the sense of Stratonovich. Hence, the statement of the corollary is an immediate consequence of Theorem 6 and the equivalence relations above. □

As in the article [**3**] a direct consequence of our last theorem are necessary and sufficient conditions for the stochastic system (f, g) to satisfy the comparison principle with respect to an arbitrary order relation in $\mathbb{R}^m$. To be more precise, let σ_1 and σ_2 be disjoint sets such that $\sigma_1 \cup \sigma_2 = \{1, \ldots, m\}$. For two vectors u and v in $\mathbb{R}^m$ we write $u \succeq v$ if

$$\begin{cases} u^j \geq v^j & \text{for } j \in \sigma_1 \\ u^j \leq v^j & \text{for } j \in \sigma_2. \end{cases}$$

Corollary 3. — *Let (f, g) be a system of stochastic PDEs. We assume that the functions g_j^l are twice continuously differentiable with respect to u^k, for all $j \in \mathbb{N}$ and $1 \leq k, l \leq m$. Then, the system (f, g) satisfies the comparison principle with respect to the order relation $\succeq$ if and only if the interaction function f satisfies*

$$\begin{cases} f^l(x, t, u) \geq f^l(x, t, v) & l \in \sigma_1 \\ f^l(x, t, u) \leq f^l(x, t, v) & l \in \sigma_2, \end{cases}$$

for $x \in O, t > 0$ and $u, v \in \mathbb{R}^m$ such that $u \succeq v$ and $u^l = v^l$, and the functions g_j^l depend on the component u^l of the solution only, that is

$$g_j^l(x, t, u^1, \ldots, u^m) = g_j^l(x, t, u^l)$$

for all $j \in \mathbb{N}$, $1 \leq l \leq m$.

Proof. — Let us define the function

$$w^j := \begin{cases} u^j - v^j & \text{if } j \in \sigma_1 \\ v^j - u^j & \text{if } j \in \sigma_2. \end{cases}$$

Then, the solutions of system (f, g) satisfy the comparison principle with respect to the order relation $\succeq$ if and only if the function w preserves positivity. By definition, w is a solution of the system $(\tilde{f}, \tilde{g})$ with

$$\tilde{f}^l(x, t, w) := \begin{cases} f^l(x, t, u) - f^l(x, t, v) & \text{if } j \in \sigma_1 \\ f^l(x, t, v) - f^l(x, t, u) & \text{if } j \in \sigma_2 \end{cases}$$

$$\tilde{g}^l(x, t, w) := \begin{cases} g^l(x, t, u) - g^l(x, t, v) & \text{if } j \in \sigma_1 \\ g^l(x, t, v) - g^l(x, t, u) & \text{if } j \in \sigma_2. \end{cases}$$

As in the proof of Theorem 6 we conclude that system $(\tilde{f}, \tilde{g})$ satisfies the positivity criterion if and only if the functions g_j^l depend on the component u^l of the solution only and the interaction term $\tilde{f}$ satisfies the positivity condition. These conditions are equivalent to the conditions on f and g stated in the theorem. □

Remark 2. — *The same result is valid when the stochastic system is interpreted in the sense of Stratonovich. This follows exactly as in the case of Theorem 6.*

The intuitive interpretation of the conditions we obtained for the stochastic perturbations is the following. In the critical case, when one component of the solution approaches zero, the stochastic perturbation needs to vanish. Otherwise, the positivity of the solution cannot be guaranteed. As for the validity of comparison principles, the critical situation occurs when the components u^l and v^l of two given solutions attain the same value. Then, the other components of the solution should have no influence on the intensity of the stochastic perturbation and the stochastic perturbations in the equation for this component of the solution necessarily coincide.

References

[1] Amann H., Invariant Sets and Existence Theorems for Semilinear Parabolic and Elliptic Systems, Journal of Mathematical Analysis and Applications, No. 65, pp. 432-467, 1978.

[2] Chueshov I.D., Vuillermot P.A., Non-random invariant sets for some systems of Parabolic stochastic partial differential equations, Stochastic Analysis and Applications Vol. 22, No. 6, pp. 1421-1486, 2004.

[3] Efendiev M., Sonner S., On verifying mathematical models with diffusion, transport and interaction, preprint, 2010.

[4] Gilbarg D., Trudinger N.S., Elliptic Partial Differential Equations of Second Order, 2nd Edition, Springer-Verlag, New York, 1983.

[5] Karatzas I., Shreve S.E., Brownian motion and stochastic calculus, 2d edition, Springer, 1991.

[6] Øksendal B., Stochastic Differential Equations: An Introduction with Applications, 6th Edition, Springer-Verlag, Berlin-Heidelberg, 2003.

[7] Stroock D.W., Varadhan S.R.S, On the support of diffusion processes with applications to the strong maximum principle, Proceedings 6th Berkeley Symposium Math. Statist. Probab., Vol. 3, University of California Press, Berkeley 1972, 333-359.

[8] Tessitore G., Zabczyk J., Wong-Zakai approximations of stochastic evolution equations, J. evol. equ. (2006), 621-655.

[9] Wong E., Zakaï M., On the relationship between ordinary and stochastic differential equations, Internat. J. Engin. Sci. 3 (1965), 213-229.

[10] Wong E., Zakaï M., Riemann-Stieljes approximations of stochastic integrals, Z. Wahrscheinlichkeitstheorie verw. Geb. 12 (1969), 87-97.

Jacky Cresson[1,2]
Messoud Efendiev[3]
Stefanie Sonner[3]