MOULD CALCULUS AND NORMALIZATION OF VECTOR FIELDS

by

Jacky CRESSON

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Contents

1. Introduction

The aim of this talk is to introduce objects of *combinatorial* nature, called *moulds* defined by Jean Ecalle in the 1970, and to discuss their use for normalization of vector fields or diffeomorphisms.

2. Normalization and continuous prenormalization

2.1. Local analytic objects. — In all the text, we consider *local analytic vector fields* on $\mathbb{C}^\nu$, $\nu \geq 1$, at 0:

$$X = \sum_{i=1}^{\nu} X_i(x)\, \frac{\partial}{\partial_{x_i}}\,,$$

$$X_i(0) = 0$$

$$X_i(x) \in \mathbb{C}\{x\} \text{ the set of analytic mapping of } \mathbb{C}^\nu$$

and *local analytic diffeomorphisms* on $\mathbb{C}^\nu$ at 0:

$$f : x \longmapsto (f_i(x), \dots, f_\nu(x))$$

$f_i(x) \in \mathbb{C}\{x\}$, $f_i(0) = 0$, $i = 1, \dots, \nu$.

Our main target is: Find "normal forms" for X and f and other *satellite objects* associated to X and f with a dynamical or analytical meaning. Many objects exist already like the *Nilpotent normal-form*, the *Poincaré-Dulac normal form*, the *correction*...etc. In the following we mainly focus on a particular class of normal form called *continuous prenormal forms* by Ecalle.

2.2. Prenormal forms. — We have a natural action on local fields or differs by change of coordinates:

$$\begin{array}{ccc} x & \xrightarrow{\quad h \quad} & h(x) = y \\ \downarrow & & \downarrow \\ X = \sum_{i=1}^{\nu} X_i(x) \frac{\partial}{\partial_{x_i}} & & \tilde{X} = \sum_{i=1}^{\nu} \tilde{X}_i(y) \frac{\partial}{\partial_{y_i}} \end{array}$$

$$X = h(Y)$$

$$(X(\tilde{\varphi} \bullet h^{-1})) \bullet h(y)\, \tilde{X}(\tilde{\varphi}(y)))$$

$$\begin{array}{ccc} \mathbb{C}\{X\} & \xrightarrow{X} & \mathbb{C}\{X\} \\ {\scriptstyle \Theta^{-1}} \uparrow & & {\scriptstyle \Theta^{-1}} \uparrow \\ \mathbb{C}\{Y\} & \xrightarrow{\tilde{X}} & \mathbb{C}\{Y\} \end{array}$$

$$\begin{array}{ccc} \varphi_x & \xleftarrow{h} & \tilde{\varphi}_{h^{-1}(x)=y} \\ X \downarrow & & \downarrow \tilde{X} \\ X(\varphi(x)) & \xleftarrow{h} & \tilde{X}(\tilde{\varphi}) \end{array}$$

conjugacy diagram.

$$\Theta \bullet X \bullet \Theta^{-1} = \tilde{X} \tag{2.1}$$

where

$$\begin{array}{cccc} \Theta : & \mathbb{C}\{x\} & \to & \mathbb{C}\{x\} \\ & \varphi & \mapsto & \varphi \bullet h \end{array} .$$

A classical question is then:

- *Can we find in all this class of associated objects special ones which are simpler?*

We will always write X or f as:

$$\begin{array}{ccc} X & = & X_{\text{lin}} + \dots \\ f & = & f_{\text{lin}} + \dots \end{array}$$

where X_{lin} and f_{lin} are the linear part of X and f respectively.

As X_{lin} (f_{lin}) is of fundamental interest from the dynamical point of view, we look for change of coordinates tangent to the identity Id:

$$h(x) = x + \underbrace{\dots\dots}_{\text{higher order terms}} .$$

In that case, we can always find a formal change of coordinates system such that

$$\tilde{X} = X_{\text{lin}} + \sum B_n \, , \quad \left(\begin{array}{l} n_i \geq -1 \\ n \stackrel{\bullet}{=} (n_1, \dots, n_\nu) \\ \text{all } n_i \in \mathbb{Z} \\ n_i \geq 0 \text{ except possibly one equal to } -1 \end{array} \right. \tag{2.2}$$

where the B_n are *homogeneous differential operators of degree* n, i.e.

$$\forall\, m \in \mathbb{N}^\nu \, , \; B_n(x^m) = \beta_{n,m}\, x^{n+m} \quad \text{where } B_{n,m} \in \mathbb{C} \, ,$$

and the operators are made of *resonant monomials*, i.e.

$$n \cdot \lambda = 0$$

where $\cdot$ is the usual scalar product. A convenient, intrinsic way to characterize the field $\tilde{X}$ is to say that one must have

$$[X_{\text{lin}}, \tilde{X}] = 0 \, .$$

A decomposition of the form (2.2) is a *prenormal form* in Ecalle's terminology. Of course, prenormal forms are not *unique*.

Between all these prenormal forms we can look for the one which have the *minimal* number of resonant parts with coefficients being *formula invariants*: this particular prenormal form is called the *normal form* X_{nor} (f_{norm}).

Unfortunately, the map

$$X \to X_{\text{nor}}$$

is not *continuous* even if the linear part is kept fixed, and moreover, their explicit computations can be carried only in a limited number of cases. This is due in part to the fact that we must prove that certain expressions depending on the Taylor coefficient of the field vanish or not, a question that a computer can't decide. We refer to the introduction of the article of Baider [**2**] for a definition of normal form and the algorithmic underlying setting.

We then look for more particular prenormal forms which will be explicitly computable in an algorithmic way: *continuous prenormal forms*.

2.3. Continuous prenormal forms. — The definition of continuous prenormal forms use the following classical decomposition of X:

$$X = X_{\text{lin}} + \sum_{n \in A(X)} B_n \tag{2.3}$$

where the B_n are homogeneous differential operators of degree n and order 1

$$B_n = x^n \left(\sum_{i=1}^{\nu} \alpha_i \, x_i \, \frac{\partial}{\partial_{x_i}} \right), \quad \alpha_i \in \mathbb{C}$$

$n = (n_1, \dots, n_\nu)$ with all $n_i \geq 0$ except one which can be equal to -1.

Example.

$$X = \lambda_1 \, x \, \partial_x + \lambda_2 \, y \, \partial_y + (a_{20} \, x^2 + a_{11} \, xy + a_{02} \, y^2) \partial_x + (b_{20} \, x^2 + b_{11} \, xy + b_{02} \, y^2) \partial_y$$

$$X_{\text{lin}} = \lambda_1 \, x \, \partial_x + \lambda_2 \, y \, \partial_y$$

$$\left[\begin{array}{lcl} B_{1,0} & = & x(a_{20} \, x \, \partial_x + b_{11} \, y \, \partial_y) \\ B_{0,1} & = & y(a_{11} \, x \, \partial_x + b_{02} \, y \, \partial_y) \\ B_{-1,2} & = & a_{02} \, y^2 \, \partial_x \\ B_{2,-1} & = & b_{20} \, x^2 \, \partial_y \, . \end{array} \right.$$

We denote by $A(X)$ the set of degrees n which arise in the decomposition (2.3).

***Definition 1* (informal).** — *A continuous prenormal form is a prenormal form which is continuous with respect to the* $\{B_n\}_{n\in A(X)}$*, the linear part being fixed.*

Note that as long as we have not defined what notion of continuity is used here, this definition is not complete.

The basic problem is now to construct continuous prenormal forms. This is what we do in the following using non-commutative formal power series.

2.4. Using enveloping algebras. — In general, we search for such objects using the Lie algebraic structure associated to $\{B_n\}_{n\in A(X)}$, and we look for

$$X_{\text{pran}} = X_{\text{lin}} + \sum_{\boldsymbol{n}\in A(X)^*} C^{\boldsymbol{n}}\, B_{[\boldsymbol{n}]} \tag{2.1}$$

where $C^{\boldsymbol{n}} \in \mathbb{C}$, $B_{[\boldsymbol{n}]} = [\dots[B_{n_1}, B_{n_2}], \dots, B_{n_{nm}}]$, $[\ ,\ ]$ the classical Lie bracket, and the *normalizator* Θ is search as an exponential of a vector field

$$\Theta = \exp\left(\sum_{\boldsymbol{n}\in A(X)^*} \Theta^{\boldsymbol{n}}\, B_{[\boldsymbol{n}]}\right), \quad \Theta^{\boldsymbol{n}} \in \mathbb{C}\,. \tag{2.2}$$

¿From the point of view of *algebraic structures* this is the *good way* to do. However, from the combinatorial point of view you must take care of the Lie structure and this induce difficulties.

A way to deal these two objects on the same setting is to "forget" the Lie structure in a first step, working in the framework of *envelopping algebra*, i.e. for us the set of *formal non-commutative series* on the set $\{B_n\}_{n\in A(X)}$, i.e. objects of the form

$$\sum_{\boldsymbol{n}\in A(X)^*} M^{\boldsymbol{n}}\, B_{\boldsymbol{n}}\,.$$

2.5. A definition. — Using the previous formalism, Jean Ecalle and Bruno Vallet [**6**] propose to look for *continuous prenormal form* given by

$$X_{\text{pran}} = X_{\text{lin}} + \sum_{\boldsymbol{n}\in A(X)^*} \text{Pran}^{\boldsymbol{n}}\, B_{\boldsymbol{n}} \tag{2.3}$$

where $A(X)^*$ is the set of sequences that we can build from $A(X)$, i.e. an element of $A(X)^*$ is a word denoted by $\boldsymbol{n}$ and of the form $\boldsymbol{n} = n_1 \dots n_r$, where r is the length of the word $\boldsymbol{n}$, and each letter $n_i \in A(X)$.

- $\mathrm{Pran}^{\boldsymbol{n}} \in \mathbb{C} \qquad \forall\, \boldsymbol{n} \in A(X)^*$
- $B_{\boldsymbol{n}} = B_{n_1} \dots B_{n_r}$ where the dot $\cdot$ stand for the classical composition of operators.

Of course, a general differential operator or (2.3) has no chance to be a vector field if we do not impose constraints on $\mathrm{Pran}^{\boldsymbol{n}}$. A part of Ecalle's work is based on the study of the specific symmetries we can put in order to obtain a vector field.

Moreover, in order to obtain a prenormal form we must satisfy the basic constraint

$$\forall\, \boldsymbol{n} \in A(X)^* \text{ such that } \|\boldsymbol{n}\cdot\boldsymbol{\lambda}\| \neq 0\,,\ \mathrm{Pran}^{\boldsymbol{n}} = 0$$

$$\|\boldsymbol{n}\cdot\boldsymbol{\lambda}\| = n_1 \cdot \boldsymbol{\lambda} + \dots + n_r \cdot \boldsymbol{\lambda}\,,\ n_i \in A(X)$$

which ensures that we only have resonant terms.

We then are leaded to the following definition of *continuous prenormal forms*:

Definition 2. — *Let X be a vector field of C^ν in prepared form given by*

$$X = X_{\mathrm{lin}} + \sum_{n \in A(X)} B_n. \tag{2.4}$$

A continuous prenormal form X_{pran} for X is a derivation of $\mathbb{C}\{x\}$ of the form

$$X_{\mathrm{pran}} = X_{\mathrm{lin}} + \sum_{\boldsymbol{n} \in A(X)^*} Pran^{\boldsymbol{n}}\, B_{\boldsymbol{n}} \tag{2.5}$$

and satisfying

$$\forall\, \boldsymbol{n} \in A(X)^* \text{ such that } \|\boldsymbol{n}\cdot\boldsymbol{\lambda}\| \neq 0\,,\ \mathrm{Pran}^{\boldsymbol{n}} = 0. \tag{2.6}$$

Using the envelopping algebra framework we can look for the *normalizator* Θ as an automorphism of $\mathbb{C}\{x\}$ of the form

$$\Theta = \sum_{\boldsymbol{n} \in A(X)^*} \Theta^{\boldsymbol{n}}\, B_{\boldsymbol{n}}. \tag{2.7}$$

As $B = \mathrm{Id}$ and the automorphism Θ must be tangent to identity we deduce that $\Theta = 1$.

Here again, a formal power serie like (2.7) is in general not an automorphism of $\mathbb{C}\{x\}$. As a consequence, the coefficients $\{\Theta^{\boldsymbol{n}}\}_{n \in A(X)^*}$ must satisfy specific symmetries.

Remark 1. — *Of course, we can try to look for continuous prenormal forms with a different dependence, i.e. that the form of Θ and X_{pran} is a choice.*

It is not clear at this point that we have come off forgetting the Lie algebraic structure. Next section is devoted to the hold problem of linearization of vector fields or diffeomorphisms. We will see that the previous formalism allows us to obtain *universal* objects associated to the linearization problem. Up to now, no other method has been able to recover this result. Moreover, vector fields and diffeomorphisms are studied using exactly the same framework and only differ by small computational details.

3. Linearization of vector fields and diffeomorphisms

3.1. Product of non-commutative formal power series. — Let

$$S=\sum_{\boldsymbol{n}\in A(X)^*} S^{\boldsymbol{n}}\,B_{\boldsymbol{n}} \quad\text{and}\qquad U=\sum_{\boldsymbol{n}\in A(X)^*} U^{\boldsymbol{n}}\,B_{\boldsymbol{n}}$$

be two non-commutative formal power series, then the product $\times$ of S and U is defined by

$$S\times U=\sum_{\boldsymbol{n}\in A(X)^*}\left(\sum_{\boldsymbol{n}_1\,\boldsymbol{n}_2=\boldsymbol{n}} S^{\boldsymbol{n}_1}\,U^{\boldsymbol{n}_2}\right)B_{\boldsymbol{n}}$$

where the summation is done under all the possible partitions of $\boldsymbol{n}$ in two sequences as

$$\boldsymbol{n}=\underbrace{n_1,\dots}_{\boldsymbol{n}_1}\underbrace{,n_r}_{\boldsymbol{n}_2}.$$

3.2. The case of vector fields. —

3.2.1. Linearization equation. — Using (2.3) and (2.7), we can easily write the linearization equation (i.e. the case $X_{\text{pran}}=X_{\text{lin}}$). We have:

$$X=\Theta^{-1}X_{\text{lin}}\,\Theta\Leftrightarrow$$

$$\sum_{n\in A(X)} B_n+X_{\text{lin}}=\left(\sum_{\boldsymbol{n}\in A(X)^*}\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}\right)^{-1}\left(X_{\text{lin}}\right)\left(\sum_{\boldsymbol{n}\in A(X)^*}\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}\right).$$

For the moment, we assume that $\left(\sum_{\boldsymbol{n}\in A(X)^*}\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}\right)^{-1}$ can be computed as

$$\Theta^{-1}=\sum_{\boldsymbol{n}\in A(X)^*}\not\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}.$$

Then, we obtain

$$\sum_{n\in A(X)} B_n+X_{\text{lin}}=\left(\sum_{\boldsymbol{n}\in A(X)^*}\not\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}\right)\left(X_{\text{lin}}\right)\left(\sum_{\boldsymbol{n}\in A(X)^*}\Theta^{\boldsymbol{n}}\,B_{\boldsymbol{n}}\right).$$

We apply the usual *product for non-commutative formal power series* rewriting

$$\sum_{n\in A(X)} B_n \qquad\text{as}\qquad \sum_{\boldsymbol{n}\in A(X)^*} I^{\boldsymbol{n}}\,B_{\boldsymbol{n}}$$

$$\text{with } I^{\phi}=0\,,\ I^{n}=1\quad\forall\,n\in A(X)\,,\ I^{\boldsymbol{n}}=0\quad\forall\,\boldsymbol{n}\,,\ \ell(\boldsymbol{n})\geq 2\,,\ \ell\text{ the length.}$$

We obtain:

Notation. *I will avoid rewriting* $\boldsymbol{n} \in A(X)^*$ *by putting a* $\bullet$. *For example a serie* $\sum_{\boldsymbol{n} \in A(X)^*} M^{\boldsymbol{n}} B_{\boldsymbol{n}}$ *is written* $\sum_{\bullet} M^{\bullet} B_{\bullet}$.

We must be able to know what is the action of X_{lin} on a series of the form $\sum_{\bullet} M^{\bullet} B_{\bullet}$.

We have:

Lemma 1. — *For any non-commutative formal power series* $\sum_{\bullet} M^{\bullet} B_{\bullet}$, *we have*

$$X_{\text{lin}} \left(\sum_{\bullet} M^{\bullet} B_{\bullet} \right) = \sum_{\bullet} \nabla M^{\bullet} B_{\bullet} + \sum_{\bullet} M^{\bullet} B_{\bullet} X_{\text{lin}}, \tag{3.1}$$

where

$$\nabla (M^{\bullet})^{\boldsymbol{n}} = (\boldsymbol{\lambda} \cdot \boldsymbol{n}) \cdot M^{\boldsymbol{n}}. \tag{3.2}$$

Proof. —

$$B_n : x^n \left(\sum_{i=1}^{\nu} \alpha_i \, x_i \, \partial_{x_i} \right)$$

$$B_n(x^m) = x^{n+m} \underbrace{\sum_{i=1}^{\nu} \alpha_i \, m_i}_{\beta_m^r}$$

$$\varphi = \sum_m a_m \, x^m \; B_n(\varphi) = \sum_m a_m \, \beta_m^n \, x^m$$

$$B_{\boldsymbol{n}=n_1 \dots n_r}(\varphi) = \sum_m a_m \, \beta_m^{n_r} \, \beta_{m+n_r}^{n_{r-1}} \dots \beta_{m+n_r+\dots+n_2}^{n_1} \, x^{m+n_1+\dots+n_r}$$

$$X_{\text{lin}}(x^m) = (\boldsymbol{\lambda} \cdot m) \, x^m$$

$$\begin{aligned} X_{\text{lin}}(B_{\boldsymbol{n}}(\varphi)) &= \sum_m a_m \, \beta_{\bullet}^{\bullet} (\boldsymbol{\lambda} \cdot m + \boldsymbol{\lambda} \cdot (n_1 + \dots + n_r)) \, x^{m+n_1+\dots+n_r} \\ &= [\boldsymbol{\lambda} \cdot (n_1 + \dots + n_r)] \cdot B_{\boldsymbol{n}}(\varphi) + B_{\boldsymbol{n}}(X_{\text{lin}}(\varphi)) \, . \end{aligned}$$

□

Hence, we have

$$\sum_{\bullet} I^{\bullet} B_{\bullet} + X_{\text{lin}} = \sum_{\bullet} \not{\Theta}^{\bullet} \times \nabla \, \Theta^{\bullet} \, B_{\bullet} + \underbrace{\sum_{\bullet} \not{\Theta}^{\bullet} \times \Theta^{\bullet} \, B^{\bullet} \, X_{\text{lin}}}_{= X_{\text{lin}}}$$

because

$$\mathrm{id} = \not\Theta \times \Theta = \sum_\bullet (\underbrace{\not\Theta^\bullet \times \Theta^\bullet}_{\substack{=1 \text{ if } \bullet=\phi \\ 0 \text{ otherwise}}}) B_\bullet \,.$$

Then we obtain:

$$\sum_\bullet I^\bullet B_\bullet = \sum_\bullet (\not\Theta^\bullet \times \nabla\Theta^\bullet) B_\bullet \,.$$

Theorem 1. — *The linearization equation is equivalent to the following equation on coefficients:*

$$I^\bullet = \not\Theta^\bullet \times \nabla\Theta^\bullet \qquad \forall \bullet \in A(A)^* \,. \tag{3.3}$$

The main interest of equation (3.3) is that it can be solved by induction on the length of $\boldsymbol{n}$ as we will see in the following.

3.2.2. The Poincaré theorem. — Using (3.3) we obtain the following recursive relation on the coefficients $\Theta^\bullet$:

$$(3.3) \Leftrightarrow \Theta^\bullet \times I^\bullet = \nabla\Theta^\bullet \,.$$

For $\bullet = \phi$, we have by assumption that $\Theta^\phi = 1$.

$$\ell = 1 \qquad n \qquad \Theta^n \underbrace{I^\phi}_{\substack{\| \\ 0}} + \underbrace{\Theta^\phi}_{\substack{\| \\ 1}} \underbrace{I^n}_{\substack{\| \\ 1}} = (n\cdot\lambda)\,\Theta^n$$

$$\Theta^n = \tfrac{1}{n\cdot\lambda} \quad \text{as long as } n\cdot\lambda \neq 0 \,.$$

$$\begin{array}{lrcll} (\boldsymbol{n}\cdot\boldsymbol{\lambda}) & \Theta^{\boldsymbol{n}} & = & \Theta^{\boldsymbol{n}^{<r}} & \boldsymbol{n} = n_1 \dots n_r \\ & \Theta^{\boldsymbol{n}} & = & \frac{1}{\boldsymbol{n}\cdot\boldsymbol{\lambda}}\,\Theta^{\boldsymbol{n}^{<r}} & \boldsymbol{n}^{<r} = n_1 \dots n_{r-1} \,. \end{array}$$

As a consequence, we have the following classical result of Poincaré on the formal linearization of vector fields under a non-resonance condition.

***Theorem 2* (Poincaré theorem)**. — *Let X be a vector field in prepared form satisfying a non-resonance condition, i.e.*

$$\boldsymbol{\lambda}.\parallel \boldsymbol{n} \parallel \neq 0 \quad \forall \boldsymbol{n} \in A^*(X), \tag{3.4}$$

where

$$\parallel \boldsymbol{n} = n_1 \dots n_r \parallel = n_1 + \dots + n_r. \tag{3.5}$$

Then X can be (formally) linearized using the automorphism denoted Θ and given by

$$\Theta = \sum_{\boldsymbol{n} \in A^*(X)} \Theta^{\boldsymbol{n}} B_{\boldsymbol{n}}, \tag{3.6}$$

where $\Theta^{\boldsymbol{n}}$ is defined by

$$\Theta^{\boldsymbol{n}} = \frac{1}{\omega_1 \cdot (\omega_1 + \omega_2) \dots (\omega_1 + \dots + \omega_r)} \tag{3.7}$$

where $\omega_i = n_i \cdot \boldsymbol{\lambda}$

The classical way to prove the Poincaré theorem (see for example [**1**]) does not allow to obtain the coefficients $\Theta^{\boldsymbol{n}}$ which only depend on the spectrum of the linear part of X and the *nature* of the non-linear component, *i.e.* on the alphabet $A(X)$. As a consequence, they can be considered as *universal* for the linearization problem. We discuss in details this notion of universality in the next section.

3.2.3. Universal coefficient of linearization. —

a) Formal non-commutative series constructed on $\{B_n\}_{n \in A(X)}$ do not enter at the end in the linearization problem. We have only recursive relations between coefficients of these series: these coefficients are what J. Ecalle calls *moulds*.

b) The exact shape of the initial vector field (for example, the exact *value* of the coefficients determining the non-linear part) is encoded by the operators part $B_{\boldsymbol{n}}$, $\boldsymbol{n} \in A(X)$. As a consequence, we have the following fundamental remark:

Let

$$\begin{aligned} X_1 &= X_{\text{lin}} + \sum_{n \in A(X_1)} B_n^1 \\ X_2 &= X_{\text{lin}} + \sum_{n \in A(X_1)} B_n^2 \end{aligned}$$

two vector fields with the same linear part and defining the same set of degrees $A(X_1) = A(X_2)$. Then the linearization mould is *exactly the same.*

But, we can also see that if X_1 and X_2 have two different linear parts then the linearization mould has the *same shape* except that we change $\omega_i^1 = n_i \cdot \lambda_1$ by $\omega_i^2 = n_i \cdot \lambda_2$ where λ_1 and λ_2 are the spectrum of X_1 and X_2 respectively.

The mould $\Theta^{\bullet}$ is then *universal* in nature.

We can formalize the previous idea using a one-parameter family of maps.

***Theorem 3* (Universality).** — *Let $\boldsymbol{La} = \{\mathrm{La}_r\}_{r \geq 1}$ be the set of $\mathbb{C}$-valued functions $\mathrm{La}_r : \mathbb{C}^r \to \mathbb{C}$ defined by*

$$\mathrm{La}_r(x_1, \dots, x_r) = \frac{1}{(x_1 + \dots + x_r)(x_1 + \dots + x_{r-1}) \dots x_1}. \tag{3.8}$$

for all $(x_1, \dots, x_r) \in \mathbb{C}^r \setminus Sa_r$, *where the singular set* Sa_r *is defined by*

$$\mathrm{Sa}_r = \{x_1 = 0\} \bigcup \{x_1 + x_2 = 0\} \bigcup \cdots \bigcup \{x_1 + \cdots + x_r = 0\}. \tag{3.9}$$

If X *is non-resonant then it can be formally linearized using an automorphism* Θ *of the form where* $\Theta^{\boldsymbol{n}}$ *is defined by*

$$\Theta^{n_1 \dots n_r} = \mathrm{La}_r(\omega_1, \dots, \omega_r), \tag{3.10}$$

where $\omega_i = n_i \cdot \boldsymbol{\lambda}$.

We can certainly go further using the terminology and formalism of categories [**8**], in particular formulating the previous universality property in a functorial way.

3.3. The case of diffeomorphisms. — The same can be done with diffeomorphism tangent to identity:

$$F = F_{\mathrm{lin}} \left(1 + \sum_{n \in A(F)} B_n \right)$$

where B_n are homogeneous differential operators of degree n. This comes from the *classical Taylor formula* (expansion)

$$F_{\mathrm{lin}} : \varphi \longmapsto \varphi(e^{\lambda_1} x_1, \dots, e^{\lambda_\nu} x_\nu)$$

where $\{e^{\lambda_i}\}_{i=1,\dots,\nu}$ is the spectrum of the linear part of f.
The linearization problem is equivalent to:

$$\left(\sum \not{\Theta}^{\bullet} B_{\bullet} \right) F_{\mathrm{lin}} \left(\sum \Theta^{\bullet} B_{\bullet} \right) = F_{\mathrm{lin}} \left(1^{\bullet} + \sum I^{\bullet} B_{\bullet} \right)$$

where $1^{\bullet} = \begin{cases} 1 & \text{if } \bullet = \phi \\ 0 & \text{otherwise.} \end{cases}$
The action of F_{lin} can also be derived by a simple computation. We have:

Lemma 2. — *For all* $\boldsymbol{n} \in A(X)^*$, *we have:*

$$B_{\boldsymbol{n}}(F_{\mathrm{lin}}(x^m)) = F_{\mathrm{lin}}(e^{-\boldsymbol{\lambda} \cdot (\boldsymbol{n}_1 + \cdots + \boldsymbol{n}_r)} B_{\boldsymbol{n}}(x^m)) \,.$$

Proof. — We have

$$B_{\boldsymbol{n}} = B_{n_1} \dots B_{n_r},$$

and

$$B_{n^i}(x^m) = \beta_m^{n_i} x^{m+n_i}.$$

As a consequence, we obtain

$$B_{\boldsymbol{n}}(x^m) = \beta_{m+n_r+\cdots+n_2}^{n_1} \beta_{m+n_r+\cdots+n_3}^{n_2} \dots \beta_m^{n_r} x^{m+n_1+\cdots+n_r}.$$

As

$$F_{\mathrm{lin}}(x^m) = e^{\lambda\cdot m}\, x^m,$$

we deduce that

$$\begin{aligned} B_{\boldsymbol{n}}(F_{\mathrm{lin}}(x^m)) &= e^{\lambda\cdot m}\, B_{\boldsymbol{n}}(x^m) \\ &= e^{-\lambda\cdot(n_1+\cdots+n_r)}\, \underbrace{e^{\lambda\cdot(m+n_1+\cdots+n_r)}\, B_{\boldsymbol{n}}(x^m)}_{F_{\mathrm{lin}}(B_{\boldsymbol{n}}(x^m))} \\ &= F_{\mathrm{lin}}(e^{-\lambda\cdot(n_1+\cdots+n_r)}\, B_{\boldsymbol{n}}(x^m))\,. \end{aligned}$$

This concludes the proof. □

We introduce a new operator acting on coefficients of non-commutative formal power series.

Definition 3. — *We denote by* e^{∇} *the operator defined by*

$$e^{-\nabla}(M^\bullet)^{\boldsymbol{n}} = e^{-\|\boldsymbol{\lambda}\cdot\boldsymbol{n}\|}\, M^{\boldsymbol{n}} \quad \forall\, \boldsymbol{n} \in A(X)^*\,. \tag{3.11}$$

Following the same computations, we obtain:

Theorem 4. — *The linearization equation for diffeomorphisms is equivalent to*

$$e^{-\nabla}\, \not\Theta^\bullet \times \Theta^\bullet = 1^\bullet + I^\bullet\,. \tag{3.12}$$

In the same way, you can compute explicitly the coefficient $\not\Theta^\bullet$ and $\not\Theta$. We obtain:

$$e^{-\nabla}\, \not\Theta^\bullet = (1^\bullet + I^\bullet) \times \not\Theta^\bullet\,.$$

As usual: $\not\Theta^\phi = 1$.

For $\ell(\boldsymbol{n}) = 1$,

$$e^{-\lambda\cdot n}\, \not\Theta^n = (\underbrace{1^n}_{\substack{\| \\ 0}} + \underbrace{I^n}_{\substack{\| \\ 1}})\, \not\Theta^\phi + (\underbrace{1^\phi}_{\substack{\| \\ 1}} + \underbrace{I^\phi}_{\substack{\| \\ 0}})\, \not\Theta^n = 1 + \not\Theta^n\,.$$

Then,

$$\not\Theta^n = \frac{1}{e^{-\lambda\cdot n} - 1}\,.$$

We have more generally an iterative formula:

$$\begin{aligned} &\boldsymbol{n} = n_1, \dots, n_r \\ &\underbrace{e^{-\nabla}\, \not\Theta^{\boldsymbol{n}=n_1\dots n_r}}_{\substack{\| \\ e^{-\boldsymbol{n}\cdot\lambda}\not\Theta^{\boldsymbol{n}}}} &&= \Big((\underbrace{1^{\boldsymbol{n}^{<i}} + I^{\boldsymbol{n}^{<i}}}_{\substack{\| \\ 0}})\, \not\Theta^{\boldsymbol{n}^{\geq i}} \Big)_{i=2,\dots,r} \\ & &&+ \ (1^{n_1} + I^{n_1})\, \not\Theta^{n_2\dots n_r} + (1^\phi + I^\phi)\, \not\Theta^{\boldsymbol{n}} \\ & &&= \ \not\Theta^{n_2\dots n_r} + \not\Theta^{\boldsymbol{n}} \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{n}^{<i} &= n_1 \dots n_{i-1} \\ \boldsymbol{n}^{\geq i} &= n_i \dots n_r \,. \end{aligned}$$

Hence,

$$\Theta^{\boldsymbol{n}} = \frac{\Theta^{n_2 \dots n_r}}{e^{-\|\boldsymbol{n}\cdot\lambda\|} - 1}$$

where $\|\boldsymbol{n} \cdot \lambda\| = n_1 \cdot \lambda + n_2 \cdot \lambda + \cdots + n_r \cdot \lambda$.

We obtain finally:

The *universal* coefficient of linearization for diffeomorphisms is:

$$\Theta^{\boldsymbol{n}} = \frac{1}{(e^{-(\omega_1+\cdots+\omega_r)} - 1)(e^{-(\omega_2+\cdots+\omega_r)} - 1)\dots(e^{-\omega_r} - 1)}$$

where $\omega_i = n_i \cdot \lambda$.

3.4. Remarks and comments. —

c) Looking for *bifurcation problems* (like the *centre problem*) we have a natural dichotomy between the mould part which is universal and the operator part $B_{\boldsymbol{n}}$, $\boldsymbol{n} \in A(X)$ which contains all the depency of the serie with respect to the non-linear terms of the field. The $B_{\boldsymbol{n}}$, $\boldsymbol{n} \in A(X)$ are called *comoulds* by Ecalle.

d) We have not discussed the *convergence* of the normalization. This can be done in the mould setting using a specific operation on moulds called the *arborification*. This will not be discussed in this lecture, but we refer to [**3**] and [**?**].

4. The mould formalism

4.1. Definition and properties. — All these computations lead us to introduce the following object:

Definition.

Let A be an alphabet (finite or not).

A^* *denote the set of words on* A.

$\boldsymbol{a} \in A^*, \qquad \boldsymbol{a} = a_1, \ldots, a_r.$

Let K be a ring (or a field).

A mapping from A^* to K, denoted $M^\bullet$

$$\begin{array}{cccc} M^\bullet & : & A^* & \rightarrow & K \\ & & \boldsymbol{a} & \mapsto & M^{\boldsymbol{a}} \end{array}$$

is a mould.

This formal definition is equivalent to the informal statement of J. Ecalle: "A mould is a function depending on a variable number of variables".

The prenormal forms and normalizators that we have constructed can be seen as follows:

Pram$^\bullet$ and $\Theta^\bullet$ the corresponding mould of prenormalization and the normalizator:

$$S_{\text{Pran}} = \sum_{\boldsymbol{n} \in A(X)^*} \text{Pran}^{\boldsymbol{n}}\, \boldsymbol{n} \qquad S_\Theta = \sum_{\boldsymbol{n} \in A(X)^*} \Theta^{\boldsymbol{n}}\, \boldsymbol{n}$$

the corresponding *generating series.*

So, to each mould on $M^\bullet$ we can associate its generating serie

$$\Phi_M = \sum_{\boldsymbol{a}} M^{\boldsymbol{a}}\, \boldsymbol{a} = \sum_{\bullet} M^\bullet\ \bullet\ .$$

Note that Φ_M belongs to $K \ll A \gg$, the set of *formal non-commutative series.*

Formal non-commutative series possess a natural structure of algebra:

i)

$$\Phi_M + \Phi_N = \sum_{\boldsymbol{a}} (M^{\boldsymbol{a}} + N^{\boldsymbol{a}})\, \boldsymbol{a}$$

ii)

$$\Phi_M \times \Phi_N = \sum_{\boldsymbol{a}} \left(\sum_{\boldsymbol{a}^1 \boldsymbol{a}^2 = \boldsymbol{a}} M^{\boldsymbol{a}^1} N^{\boldsymbol{a}^2} \right) \boldsymbol{a}\, .$$

These operations induce a natural structure of algebra on the set of moulds, which we denote by $\mathcal{M}_K(A)$:

i) $\Rightarrow$

$$M^\bullet + N^\bullet = S^\bullet$$

is defined by $\forall\, \boldsymbol{a} \in A^*$,

$$S^{\boldsymbol{a}} = M^{\boldsymbol{a}} + N^{\boldsymbol{a}}\,.$$

ii) $\Rightarrow$

$$M^\bullet \times N^\bullet = P^\bullet$$

is defined by $\forall\, \boldsymbol{a} \in A^*$,

$$P^{\boldsymbol{a}} = \sum_{\boldsymbol{a}^1 \boldsymbol{a}^2 = \boldsymbol{a}} M^{\boldsymbol{a}^1}\, N^{\boldsymbol{a}^2}\,.$$

There exists another operation, which will not be used in this lecture, called "composition", which is the non-commutative analogue of the classical *substitution* operation for classical formal power series.

4.2. Lie structure and connexion to vector fields. — For all this part, we refer the reader to the book of J.-P. Serre [**11**], and the book of C. Reutenauer [**10**]. We can define a Lie bracket

$$[a_1, a_2] = a_1\, a_2 - a_2\, a_1$$

and look for the Lie algebra generated by A in $K \ll A \gg$. We denote it by $\mathcal{L}_A$. We have

$$\mathcal{L}_A \subset K \ll A \gg$$

and a natural mapping from $\mathcal{L}_A$ to $K \ll A \gg$:

$$\begin{array}{cclcl} \iota & : & \mathcal{L}_A & \hookrightarrow & K \ll A \gg \\ & & a_i & \mapsto & a_i \\ & & [a_i, a_j] & \mapsto & a_i\, a_j - a_j\, a_i\,. \end{array}$$

The Lie algebra $\mathcal{L}_A$ is called the *free-Lie algebra* and $K \ll A \gg$ is isomorphic to the *envelopping algebra* of $\mathcal{L}_A$.

The problem is now to characterize the Lie element in $K \ll A \gg$. This can be done using what is called a *coproduct*, i.e. a mapping $\Delta : K \ll A \gg \to K \ll A \gg \otimes K \ll A \gg$ which is a morphism and defined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x \qquad \forall\, x \in A\,.$$

[I recall that the tensor product is defined as follows:

$$(a \otimes b) \cdot (c \otimes d) = ac \otimes bd\,.]$$

The following theorem characterizes the Lie elements:

Theorem.

$$\mathcal{L}_A = \{\Phi \in K \ll A \gg\,,\ \Delta\Phi = \Phi \otimes 1 + 1 \otimes \Phi\}\,.$$

An element satisfying $\Delta\, x = x \otimes 1 + 1 \otimes x$ is called *primitive.*

At this point it is perhaps better to look for our particular underlying problem of normalization for vector fields.

Vector fields on $\mathbb{C}\{x\}$ (or $\mathbb{C} \ll x \gg$) are *derivations* on the algebra $\mathbb{C}\{x\}$, i.e.:

i) X is a $\mathbb{C}$-linear mapping on $\mathbb{C}\{x\}$ and

ii) $X(f \cdot g) = X(f) \cdot g + f \cdot X(g)$

which generalizes the classical *Leibniz rule* ($(fg)' = f'g + fg'$) for functions.

The coproduct Δ on $K \ll A \gg$ can be seen as the analogue of the Leibniz rule for the B_n, $n \in A(X)$. Indeed, we look for series in $\mathbb{C} \ll \mathcal{B} \gg$, where $\mathcal{B} = \{B_n\}_{n \in A(X)}$. For each $n \in A(X)$, we have

$$B_n(f \cdot g) = B_n\, f \cdot g + f \cdot B_n\, g$$

as the B_n come from a vector field (things are different for diffeomorphisms, as we shall see later).

Let μ be the mapping

$$\begin{array}{rcccc} \mu & : & \mathbb{C}\{x\} \otimes \mathbb{C}\{x\} & \rightarrow & \mathbb{C}\{x\} \\ & & f \otimes g & \mapsto & f \times g \end{array}$$

We define the $\Delta(B_n)$ for each $n \in A$ as follows:

(4.1)
$$\begin{array}{ccc} \mathbb{C}\{x\} \otimes \mathbb{C}\{x\} & \xrightarrow{\Delta(B_n)} & \mathbb{C}\{x\} \otimes \mathbb{C}\{x\} \\ f \otimes g & \longmapsto & \Delta(B_n)(f \otimes g) \\ \downarrow \mu & & \downarrow \mu \\ \mathbb{C}\{x\} & \xrightarrow{\quad B_n \quad} & \mathbb{C}\{x\} \\ f \times g & & B_n\, f \cdot g + f \cdot B_n\, g \end{array}$$

As a consequence, one can take:

$$\Delta(B_n) = B_n \otimes 1 + 1 \otimes B_n\,.$$

With the same kind of diagramm, we have

$$\Delta(1) = 1 \otimes 1$$

where 1 stands for the identity map.

We can also defined directly $\Delta(B_{\boldsymbol{n}}) = \Delta(B_{n_1}\dots B_{n_r})$, and we verify that

$$\Delta(B_{n_1}\dots B_{n_r}) = \Delta(B_{n_1})\dots\Delta(B_{n_r})$$

so that Δ is a *morphism* of $\mathbb{C} \ll\mathcal{B}\gg$ in $\mathbb{C} \ll\mathcal{B}\gg \otimes \mathbb{C} \ll\mathcal{B}\gg$ (the K-linearity is satisfied).

In order to give the classical terminology, I must introduce the mapping $\varepsilon : \mathbb{C} \ll\mathcal{B}\gg \to \mathbb{C}$, which is $\mathbb{C}$-linear and defined by $\varepsilon\left(\Phi = \sum_{\boldsymbol{n}\in A(X)^*} M^{\boldsymbol{n}} B_{\boldsymbol{n}}\right) = M^{\phi} \in \mathbb{C}$, the constant term of Φ.

The triplet $(\mathbb{C} \ll\mathcal{B}\gg, \varepsilon, \Delta)$ is called a cogebre in Bourbaki and a *coalgebra* in general. (We refer to the book of David Eisenbud [**7**] for more details.) The main point using our coproduct Δ is that Φ is a *derivation* (then a vector field) if

$$\Delta\Phi = \Phi\otimes 1 + 1\otimes\Phi\,,$$

i.e. Φ is primitive. A formal relation with the pure algebraic part of the beginning is to replace each B_n, $n\in A(X)$ by a letter b_n, $n\in A(X)$.

We have now an alphabet $b = \{b_n\}_{n\in A(X)}$ which is formal, i.e. *free*, contrary to the $\mathcal{B}$. In general, we have some relations between the B_n, $n\in A(X)$, as for example a relation of the type $[B_{n^1}, B_{n^2}] = B_{n^3}$, $n^1, n^2, n^3 \in A(X)$.

As a consequence, replacing a given serie $\Phi = \sum_{\boldsymbol{n}\in A(X)^*} M^{\boldsymbol{n}} B_{\boldsymbol{n}}$ by its *free* counterpart $\Phi_\ell = \sum_{\boldsymbol{n}\in A(X)^*} M^{\boldsymbol{n}} b_{\boldsymbol{n}}$ allows us to work in the classical context of free-Lie algebra.

Using the same diagramm as (4.1), we also see that *automorphisms* of $\mathbb{C}\{x\}$ can be characterized via Δ, imposing that the serie Φ satisfies

$$\Delta\Phi = \Phi\otimes\Phi\,.$$

These elements are called *group-like* in the context of envelopping algebra.

A natural correspondence exists between primitive and group-like elements via the *exponential* and the *logarithm* map. We refer to [**11**] or [**10**] for more details.

4.3. Mould symmetries, vector fields and automorphisms. — As we pointed in the beginning of this chapter, this is not clear if a given serie

$$S = \sum_{\boldsymbol{n}\in A(X)^*} M^{\boldsymbol{n}} B_{\boldsymbol{n}} \tag{4.2}$$

is a vector field or an automorphism. At least, we must have very specific *symmetries* which are satisfied by the coefficients.

In the previous section, we have characterized these objects using the coproduct Δ. We obtain:

$$\Delta\, S = S \otimes 1 + 1 \otimes S \tag{4.3}$$

for a vector field, and

$$\Delta\, S = S \otimes S \tag{4.4}$$

for an automorphism.

Let us compute $\Delta(S)$ directly. We have

$$\Delta\, S = \sum_{\boldsymbol{n} \in A(X)^*} M^{\boldsymbol{n}}\, \Delta\, B_{\boldsymbol{n}}\,, \tag{4.5}$$

due to the linearity of Δ with respect to $\mathbb{C}$. Moreover,

$$\Delta\, B_n = \sum_{(\boldsymbol{n}^1, \boldsymbol{n}^2) \in \mathcal{C}_{\boldsymbol{n}}} B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2}$$

where $\mathcal{C}_{\boldsymbol{n}}$ is the set of couple of sequences which appears when we compute $\Delta\, B_{\boldsymbol{n}}$. Of course, this set must be described, and this is precisely what we do in the following.

At least, we can write for the moment,

$$\begin{aligned} \Delta\, S &= \sum_{\boldsymbol{n} \in A(X)^*} M^{\boldsymbol{n}} \sum_{(\boldsymbol{n}^1, \boldsymbol{n}^2) \in \mathcal{C}_{\boldsymbol{n}}} B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} \\ &= \sum_{\boldsymbol{n}^1, \boldsymbol{n}^2 \in A(X)^*} \left(\sum_{\boldsymbol{n} \in \mathcal{T}_{\boldsymbol{n}^1, \boldsymbol{n}^2}} M^{\boldsymbol{n}} \right) B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} \end{aligned} \tag{4.6}$$

where $\mathcal{T}_{\boldsymbol{n}^1, \boldsymbol{n}^2}$ will be described explicitly.

A formula like (4.6) will allow us to characterize directly vector fields and automorphisms.

4.3.1. Structure of the set $\mathcal{C}_{\boldsymbol{n}}$. — The set $\mathcal{C}_{\boldsymbol{n}}$ can be define by induction using the fact that: Let $n \in A(X)$, $\boldsymbol{n} = n_1 \ldots n_r$ be given

$$\Delta\, B_{n\boldsymbol{n}} = \Delta(B_{nn_1 \ldots n_r}) = \Delta\, B_n\, \Delta(B_{n \ldots n_r})$$

as Δ is a morphism.

By assumption, we have

$$\Delta\, B_n = B_n \otimes 1 + 1 \otimes B_n\,.$$

Let us assume that for all $\boldsymbol{n}$ of length $\ell(\boldsymbol{n}) \leq r$ we have described $\mathcal{C}_{\boldsymbol{n}}$ then $\mathcal{C}_{n\boldsymbol{n}}$ is obtained as follows:

$$\Delta\, B_{n\boldsymbol{n}} = (B_n \otimes 1 + 1 \otimes B_n) \sum_{(\boldsymbol{n}^1, \boldsymbol{n}^2) \in \mathcal{C}_n} B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2}\,.$$

Using the *tensor product* rule

$$(a \otimes b)\,(c \otimes d) = (ac \otimes bd)\,,$$

we obtain

$$\Delta\, B_{n\boldsymbol{n}} = \sum_{(\boldsymbol{n}^1,\boldsymbol{n}^2)\in\mathcal{C}_{\boldsymbol{n}}} B_n\, B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} + \sum_{(\boldsymbol{n}^1,\boldsymbol{n}^2)\in\mathcal{C}_{\boldsymbol{n}}} B_{\boldsymbol{n}^1} \otimes B_n\, B_{\boldsymbol{n}^2}\,.$$

As a consequence:

$$\mathcal{C}_{n\boldsymbol{n}} = \{(n\,\boldsymbol{n}^1, \boldsymbol{n}^2), (\boldsymbol{n}^1, n\,\boldsymbol{n}^2), (\boldsymbol{n}^1, \boldsymbol{n}^2) \in \mathcal{C}_n\}\,.$$

We introduce two actions on couple of words defined by: $\forall\, n \in A(X)$,

$$\begin{array}{rccc} a_n^- & : & A(X)^* \times A(X)^* & \to & A(X)^* \times A(X)^* \\ & & (\boldsymbol{n}^1, \boldsymbol{n}^2) & \mapsto & (n\,\boldsymbol{n}^1, \boldsymbol{n}^2)\,, \end{array}$$

$$\begin{array}{rccc} a_n^+ & : & A(X)^* \times A(X)^* & \to & A(X)^* \times A(X)^* \\ & & (\boldsymbol{n}^1, \boldsymbol{n}^2) & \mapsto & (\boldsymbol{n}^1, n\,\boldsymbol{n}^2)\,. \end{array}$$

Using these actions, we have

$$\mathcal{C}_{n\boldsymbol{n}} = a_n^\sigma(\mathcal{C}_{\boldsymbol{n}})\,, \qquad \sigma = \pm\,.$$

Denoting by $\mathcal{C}_\phi = (\phi, \phi)$ the empty couple, we have that

$$(\boldsymbol{n}^1, \boldsymbol{n}^2) \in \underset{n_1\ldots n_r}{\underset{\Vert}{\mathcal{C}_{\boldsymbol{n}}}} \Leftrightarrow \exists\, \sigma_1, \ldots, \sigma_r \in \{\pm\}$$

such that $(\boldsymbol{n}^1, \boldsymbol{n}^2) = a_{n_1}^{\sigma_1} \ldots a_{n_r}^{\sigma_r}(\phi, \phi)$.

4.3.2. Shuffle and structure of $\mathcal{T}_{\boldsymbol{n}^1,\boldsymbol{n}^2}$. — We then have a complete and explicit characterization of $\mathcal{C}_{\boldsymbol{n}}$. What about $\mathcal{T}_{\boldsymbol{n}^1,\boldsymbol{n}^2}$?

We fix a given couple $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ and we look for the set of $\boldsymbol{n} \in A(X)^*$ such that $(\boldsymbol{n}^1, \boldsymbol{n}^2) \in \mathcal{C}_{\boldsymbol{n}}$.

There is a canonical writing for $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ using the actions $a_\bullet^+$ and $a_\bullet^-$.

Let $\boldsymbol{n}^1 = n_1^1 \ldots n_p^1$ and $\boldsymbol{n}^2 = n_1^2 \ldots n_q^2$, then

$$(\boldsymbol{n}^1, \boldsymbol{n}^2) = a_{n_1^1}^- \ldots a_{n_p^1}^-\, a_{n_1^2}^+ \ldots a_{n_q^2}^+\,(\phi, \phi)\,. \tag{4.7}$$

Moreover, to a given decomposition $a_{n_1}^{\sigma_1} \ldots a_{n_r}^{\sigma_r}\,(\phi, \phi)$ we can associate a unique word of $A(X)^* : n_1 \ldots n_r$.

We denote by π this mapping:

$$\pi\,(a_{n_1}^{\sigma_1} \ldots a_{n_r}^{\sigma_r}\,(\phi, \phi)) = n_1 \ldots n_r\,.$$

Now we make the following remarks:

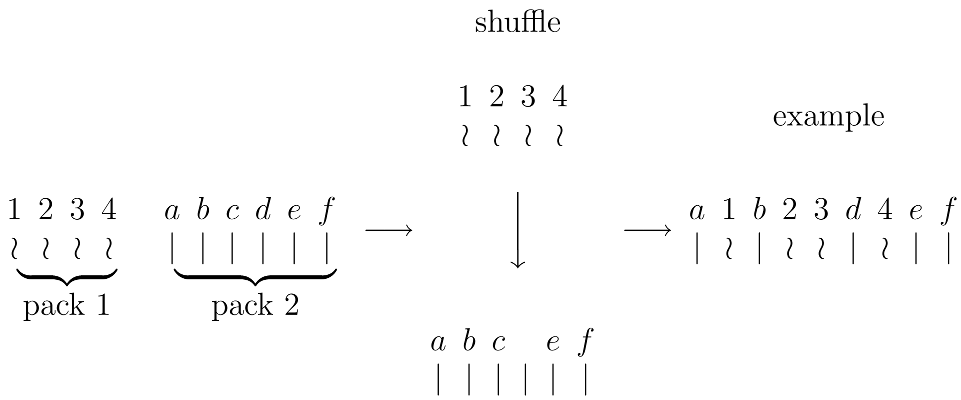
i) The operators $a_\bullet^+$ and $a_\bullet^-$ commute.

ii) The operators $a^+_\bullet$ do not commute a priori. The same for $a^-_\bullet$.

As a consequence, given the canonical decomposition (4.7), we can obtain all the $\boldsymbol{n}$ which generate $(\boldsymbol{n}^1, \boldsymbol{n}^2)$ using the rules i) and ii) and the mapping π.

This means that we can mix the letters of $\boldsymbol{n}^1$ and $\boldsymbol{n}^2$ but *preserving* the internal order of each of them (this comes from property ii): letters of $\boldsymbol{n}^1$ associated to $a^-_\bullet$ operators, so that we can not invert the order of two letter of $\boldsymbol{n}^1$ (the same for $\boldsymbol{n}^2$) except if they are the same).

This is the classical *shuffle* for cards:



The classical term in combinatoric of free Lie algebra is precisely *shuffle* and is denoted by $\sqcup\!\sqcup$. Then

$$\mathcal{T}_{\boldsymbol{n}^1,\boldsymbol{n}^2} = \boldsymbol{n}^1 \sqcup\!\sqcup\, \boldsymbol{n}^2 \,.$$

4.3.3. Symmetries of moulds. — We now return to an initial problem. Formula (4.6) is now written as:

$$\Delta S = \sum_{\boldsymbol{n}^1,\boldsymbol{n}^2 \in A(X)^*} \left(\sum_{\boldsymbol{n}\in \boldsymbol{n}^1 \sqcup\!\sqcup\, \boldsymbol{n}^2} M^{\boldsymbol{n}} \right) B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} \,. \tag{4.8}$$

We have also:

$$\Delta S = S\otimes 1 + 1 \otimes S + \sum_{\substack{\boldsymbol{n}^1,\boldsymbol{n}^2 \in A(X)^* \\ \boldsymbol{n}^1 \neq \phi \\ \boldsymbol{n}^2 \neq \phi}} \left(\sum_{\boldsymbol{n}\in \boldsymbol{n}^1 \sqcup\!\sqcup\, \boldsymbol{n}^2} M^{\boldsymbol{n}} \right) B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} \,. \tag{4.9}$$

Formula (4.9) gives us the following lemma:

Lemma. *If the mould $M^\bullet$ satisfies the symmetry*

$$\forall\, \boldsymbol{n}^1, \boldsymbol{n}^2 \in A(X)^* \,,\ \boldsymbol{n}^1 \neq \phi \,,\ \boldsymbol{n}^2 \neq \phi \qquad \sum_{\boldsymbol{n}\in \boldsymbol{n}^1 \sqcup\!\sqcup\, \boldsymbol{n}^2} M^{\boldsymbol{n}} = 0 \,, \tag{4.10}$$

then S is primitive.

Of course this is not an *equivalence* because the alphabet $A(X)^*$ is a priori *not free*. We obtain a complete equivalence for free Lie-algebra.
We can also look for automorphisms. We have

$$S \otimes S = \sum_{\boldsymbol{n}^1, \boldsymbol{n}^2 \in A(X)^*} M^{\boldsymbol{n}^1} M^{\boldsymbol{n}^2} B_{\boldsymbol{n}^1} \otimes B_{\boldsymbol{n}^2} .$$

As a consequence, using formula (4.8) we have:

Lemma. *If the mould $M^\bullet$ satisfies the symmetry*

$$\forall \boldsymbol{n}^1, \boldsymbol{n}^2 \in A(X)^* , \sum_{\boldsymbol{n} \in \boldsymbol{n}^1 ⧢ \boldsymbol{n}^2} M^{\boldsymbol{n}} = M^{\boldsymbol{n}^1} M^{\boldsymbol{n}^2} , \tag{4.11}$$

then S is group-like.

For the same reasons, as for primitive elements, this is not an equivalence.
We then have the following terminology of Ecalle:

Definition. *A mould $M^\bullet$ is said* alternal *(resp.* symetral*) if it satisfies condition* (4.10) *(resp.* (4.11)*).*

The same can be done for diffeomorphisms with more complicated symmetries (see [**3**]).

5. About the Poincaré-Dulac normal form

We have see that moulds are usefull to extract universal coefficients for normalization problems. In this part, we discuss more precisely what Ecalle call the *trimmed form* and which is constructed on the model of the *Poincaré-Dulac normal form*.

5.1. The general strategy. — Let $X = X_{\text{lin}} + \sum_{n\in A(X)} B_n$ be a field in *prepared form*, meaning that X_{lin} is diagonal and the B_n are homogeneous differential operator of degree n and order 1,

$$X_{\text{lin}} = \sum_{i=1}^{\nu} \lambda_i \, x_i \, \partial_{x_i} \, , \lambda = (\lambda_1, \dots, \lambda_\nu) \, .$$

We want to cancel the *non-resonant operators* of X. Let us denote by $N(X) \subset A(X)$ the subset of $A(X)$ consisting of *non-resonant* $n \in A(X)$, i.e.

$$n \cdot \lambda \neq 0 \, .$$

A way to do that is to look for an automorphism Θ which is given by the *exponential of a vector field*:

$$\Theta = \exp H \, , \tag{5.1}$$

where H must be computed.

Remark. Equation (5.1) must be understood as follows:

– If we work in the classical context, then exp is the Lie exponential of H.

– If we work in the mould context, this will be exp with respect to the mould product.

Of course, at the end, these two approaches define the same object.

The conjugacy equation via (5.1) is given by:

$$X_1 = \Theta \, X \, \Theta^{-1}$$

following the first chapter of these notes, which is nothing else than

$$X_1 = (\exp H) \, X \, (\exp -H) \, .$$

In the Lie context, the classical *Campbell-Baker-Hausdorff formula* gives:

$$X_1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \, X^{(n)}$$

where $X^{(n+1)} = [X^{(n)}, H]$ and $X^{(0)} = X$, then we obtain:

$$X_1 = X - [X, H] + \cdots \, .$$

Moreover, we have

$$[X,H] \;=\; \underbrace{[X_{\rm lin},H]}_{(1)} \;+\; \underbrace{[\textstyle\sum_{n\in A(X)} B_n, H]}_{(2)} .$$

Assume that H is made of differential operators of degree (of order 1) at least k, $|k| = k_1+\cdots+k_\nu \geq 1$.

Then, (2) produces operators of degree at least $|k|+|n|$, and we can not expect cancellations with terms $B_{n_\bullet}$, such that $|n_\bullet| < |n|+|k|$.

But at least, we can use the term

$$[X_{\rm lin},H] \quad \text{to cancel} \sum_{n\in N(X)} B_n .$$

This can easily be done choosing

$$H = \sum_{n\in N(X)} \frac{B_n}{n\cdot\lambda} .$$

Indeed, let

$$B_n = x^n \left(\sum_{i=1}^{\nu} a_i\, x_i\, \partial_{x_i} \right)$$

then

$$[X_{\rm lin}, B_n] = (n\cdot\lambda)\, B_n .$$

As a consequence, we have

$$X_1 = X - [X,H] = X_{\rm lin} + \sum_{n\in R(X)} B_n + \cdots$$

where $R(X)$ is the set of degrees $n\in A(X)$ which are *resonant*, i.e. $n\cdot\lambda = 0$.

The automorphism

$$\Theta = \exp\left(\sum_{n\in N(X)} \frac{B_n}{n\cdot\lambda} \right)$$

has a mould form given by

$$\Theta = \exp\left(\sum_{\boldsymbol{n}\in A(X)^*} J^{\boldsymbol{n}}\, B_{\boldsymbol{n}} \right)$$

with

$$J^{\boldsymbol{n}} = \begin{cases} \dfrac{1}{n\cdot\lambda} & \text{if } \ell(n)=1\,,\ n\in N(X) \\[2ex] 0 & \text{otherwise.} \end{cases}$$

We can write Θ in a more convenient form:

$$\Theta = \sum_{\boldsymbol{n} \in A(X)^*} (\exp J)^{\boldsymbol{n}} \, B_{\boldsymbol{n}} \,,$$

where $\exp J^\bullet$ is the *mould exponential* of $J^\bullet$ defined by

$$\exp J^\bullet = 1 + J^\bullet + \frac{1}{2} \, J^\bullet \times J^\bullet + \frac{1}{3!} \, J^\bullet \times J^\bullet \times J^\bullet + \cdots .$$

Using Θ, we obtain a *simplified* field, denoted X_{Sam} by Ecalle. By definition, X_{Sam} possesses a *mould expansion* denoted

$$X_{\mathrm{Sam}} = X_{\mathrm{lin}} + \sum_{\boldsymbol{n} \in A(X)^*} \mathrm{Sam}^{\boldsymbol{n}} \, B_{\boldsymbol{n}} \,,$$

where $\mathrm{Sam}^\bullet$ is the mould associated to this simplified form.

We then proceed by induction:

$$\begin{array}{ccc}
X & & \\
\downarrow \Theta_1 & & \\
X^1_{\mathrm{Sam}} & & \mathrm{Sam}^\bullet \\
\downarrow \Theta_2 & & \\
X^2_{\mathrm{Sam}} & & \\
\downarrow & & \\
\vdots & & \\
\downarrow & & \\
X_{\mathrm{Tram}} & & \mathrm{Tram}^\bullet
\end{array}$$

The limit of this process is called the *trimmed form* and is denoted X_{Tram}.

Again, by construction X_{Tram} possesses a mould expansion given by

$$X_{\mathrm{Tram}} = X_{\mathrm{lin}} + \sum_{\boldsymbol{n} \in A(X)^*} \mathrm{Tram}^{\boldsymbol{n}} \, B_{\boldsymbol{n}} \,.$$

The fact that this object is well defined comes from the following classical remark: at each step, the degree of the remaining non-resonant terms become more and more large. As a consequence, the coefficients of the field *stabilize* under iteration.

The main point is that we can *compute explicitly* all the moulds $\mathrm{Sam}^{\bullet}$ and $\mathrm{Tram}^{\bullet}$. As usual, they depend only on the linear part and the alphabet so that we again extract the *universal part* of this normalization procedure. (For the complete computations in the case of vector-fields we refer to [**3**].)

It is not clear that we obtain the classical Poincaré-Dulac normal[(1)] using this procedure. Indeed, the automorphism Θ is constructed using *all* the non-resonant parts.

5.2. The Poincaré-Dulac normal form using moulds. — However, if we have more than one homogeneous component in the field, this will induce *parasite-terms.* Let

$$X = X_{\text{lin}} + \underbrace{\ \dots\ }_{\substack{\text{resonant of}\\ \text{degree} \le k}} + \underbrace{\sum_{n} B_n}_{\substack{\text{non-resonant}\\ \text{degree } k}} + \underbrace{\ \dots\ }_{\substack{k+1 \text{ resonant or}\\ \text{non-resonant term}}} .$$

As we already see the bracket

$$\left[B_m, \sum_{n \in N(X)} \frac{B_n}{n \cdot \lambda} \right]$$

induces terms of degree $|m| + |n|$.

If $|n| = k$, then we have terms $|m|+k$, meaning that we will have some *interference* between terms of degree $|m| + k$ during the procedure. This means that apart from cancelling the non-resonant B_n of degree k (i.e. $|n| = k$), we are not sure to cancel all the non-resonant B_n for $|n| > k$.

This lead us to introduce the following *modification* of Ecalle's procedure, more close to *Poincaré-Dulac procedure.*

Let

$$X = X_{\text{lin}} + \sum_{k \ge 1} \underbrace{\left(\sum_{\substack{n \in A(X)\\ |n| = k}} B_n \right)}_{\mathcal{B}_k} .$$

The operator $\mathcal{B}_k$ gathers all the operators with the same degree of homogeneity:

$$B_n = x^n \left(\sum_{i=1}^{\nu} \alpha_i \, x_i \, \frac{\partial}{\partial_{x_i}} \right), \quad |n| = k$$

is an homogeneous vector field of degree $k+1$, meaning that all its components are homogeneous polynomials of degree k.

(1) In the *Hamiltonian* case, the Poincaré-Dulac normal form is the *Birkhoff normal form* which is *unique* (see for example [**9**]).

We think that the best procedure is to take:

$$X_{\text{Poin}} = \exp\left(H_{\text{Poin}}\right) X \; \exp\left(-H_{\text{Poin}}\right)$$

where

$$H_{\text{Poin}} = \sum_{\substack{n \in N(X) \\ |n| = k_0}} \frac{B_n}{n \cdot \lambda} \, ,$$

with k_0 the first degree such that the set

$$N_k = \{ n \in A(X) \, , \; |n| = k \, , \; n \cdot \lambda \neq 0 \}$$

is *non-empty.*

As a consequence, we only cancel the resonant terms of degree $k_\bullet$.

As in the previous case, all this procedure stabilizes and we obtain a normal form which is the classical Poincaré-Dulac normal form denoted here X_{Dulac}.

$$\begin{array}{ccc}
X & & \\
\big\downarrow \Theta^1_{\text{Poin}} & & \\
X^1_{\text{Poin}} & & \\
\big\downarrow \Theta^2_{\text{Poin}} & & \\
X^2_{\text{Poin}} & & \text{Poin}^\bullet \\
\big\downarrow & & \\
\vdots & & \\
\big\downarrow & & \\
X_{\text{Dulac}} & & \text{Dulac}^\bullet
\end{array}$$

All these objects have a mould expansion:

$$X_{\text{Poin}} = X_{\text{lin}} + \sum_{\boldsymbol{n} \in A(X)^*} \text{Poin}^{\boldsymbol{n}} \, B_{\boldsymbol{n}} \, ,$$

$$X_{\text{Dulac}} = X_{\text{lin}} + \sum_{\boldsymbol{n} \in A(X)^*} \text{Dulac}^{\boldsymbol{n}} \, B_{\boldsymbol{n}} \, ,$$

which can be compute *explicitly* using recursive relations on the length of $\boldsymbol{n}$.

The same procedure applies for diffeomorphisms. However, there are several *differences*, the main one being:

If we denote by X^r_{Poin} or X^r_{Sam} the resulting field after r-Poincaré normalization or simplification, we obtain a convergent vector field as well as a convergent normalizator. In the Poincaré case, this normalizator is *polynomial*.

This is *not* the case for diffeomorphisms. We refer to the paper of Ecalle-Vallet [**6**], for more details.

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Jacky CRESSON, *Université de Pau et Pays de l'Adour,, Laboratoire de Mathématiques Appliquées de Pau*, et, *I.H.É.S., 35 route de Chartres, F-91440 Bures-sur-Yvette*,

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