Advanced Mechanics

ISA-BTP 5 / M2

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2023-2024

Introduction

Non linear behaviour of structures

- Concerns the major part of it's behaviour
- Can be approximated by linear computation with asumptions $\Rightarrow$ Security coefficients.
- Non linearities due to
  - Geometrical (buckling, large strains displacements)
  - Material non linearities (plasticity, cracking)
  - Dynamic effects

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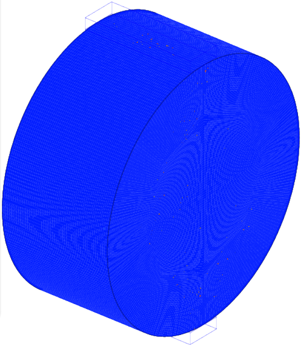

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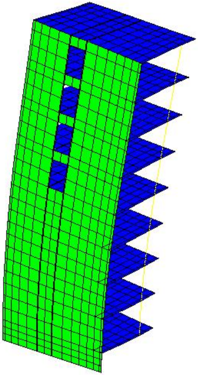

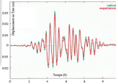

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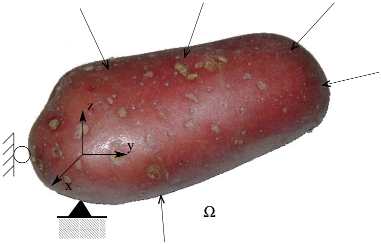
Stress

- Stresses represent the cohesive forces in a solid that allow the material to withstand the loading.
- Stresses are the result of interaction between small parts of the material (crystals, molecules... etc... etc...).
- The equivalent of stress for a perfect fluid is pressure.

Definition of the stress vector :

balance equations

- For a solid Ω loaded by a set of mechanical actions and in balance with respect to a reference system, the balance equations are verified for any arbitrary part of Ω.
- If we cut Ω by a plane of normal $\overrightarrow{n}$ passing through point P, the two parts Ω^+ located on the normal side and Ω^- located on the opposite side, are in balance.



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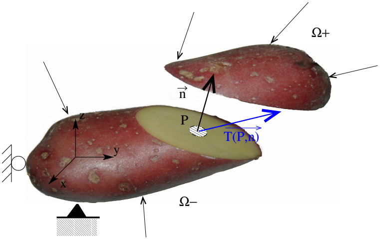
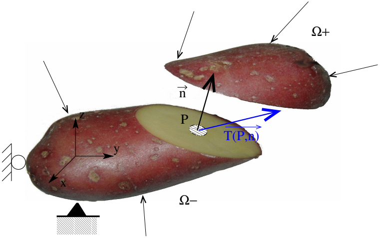


Definition of the stress vector :

Ω^- is in balance under the effect :

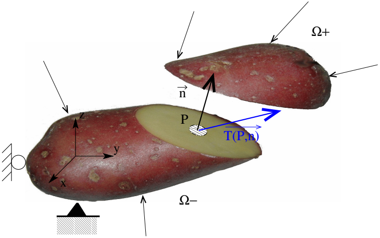
- External forces exerted on it.
- From the stress vector $\overrightarrow{T(P,\vec{n})}$ exerted at any point P of the cut-off plane.



Definition of the stress vector :

definition

The stress vector $\overrightarrow{T\left(P, \overrightarrow{n}\right)}$ is the surface density of the forces exerted by Ω^{+} on Ω^{-}. It is the physical variable associated with the stress.



Remarks

1. The stress vector is expressed in Pascals.
   $1Pa = 1N/m^2,\ 1MPa = 10^6 Pa,\ 1GPa = 10^9 Pa$
   There are also more exotic units of use strongly discouraged :
   $1T/m^2 \simeq 10kPa,\ 1kg/cm^2 \simeq 100kPa,\ 1bar = 100kPa,\ 1PSI \simeq 6,9MPa$
2. If at a given point P $T(P, \overrightarrow{n_2}) \neq T(P, \overrightarrow{n_1})$.
3. At two points P and Q of the same normal cut-off plane $T(P, \overrightarrow{n}) \neq T(Q, \overrightarrow{n})$
4. The resulting actions from Ω^+ on Ω^- for a normal cut-off plane Π of normal $\overrightarrow{n}$ is :

$$\left\{ \begin{array}{c} \overrightarrow{F_{\Omega^+/\Omega^-}} = \int\int_\Pi \overrightarrow{\overrightarrow{T(P, n)}} ds \\ \overrightarrow{M_{A,\Omega^+/\Omega^-}} = \int\int_\Pi \overrightarrow{AP} \wedge \overrightarrow{\overrightarrow{T(P, n)}} ds \end{array} \right\}_A$$

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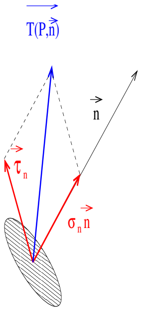
Normal and tangential stresses :

The stress vector is composed of a normal stress σ_n and a tangential stress $\overrightarrow{\tau_n}$.

$\sigma_n = \overrightarrow{T(P, \overrightarrow{n})} \bullet \overrightarrow{n}$

$\overrightarrow{\tau_n} = \overrightarrow{T(P, \overrightarrow{n})} \bullet \overrightarrow{n} - \sigma_n \overrightarrow{n}$

- The normal stress σ_n is a number.
- The tangential stress $\overrightarrow{\tau_n}$ is a vector.

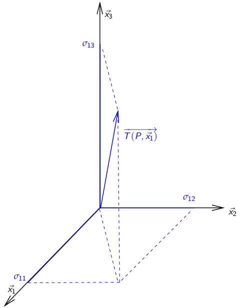


Projections on the reference vectors :

For a given reference system $(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3})$ of orthonormal vectors

- $\sigma_{11} = \overrightarrow{T(P, \overrightarrow{x_1})} \bullet \overrightarrow{x_1}$
- $\sigma_{12} = \overrightarrow{T(P, \overrightarrow{x_1})} \bullet \overrightarrow{x_2}$
- $\sigma_{13} = \overrightarrow{T(P, \overrightarrow{x_1})} \bullet \overrightarrow{x_3}$

$$\sigma_{ij} = \overrightarrow{T(P, \overrightarrow{x_i})} \bullet \overrightarrow{x_j}$$



Projections of $\overrightarrow{T(P, \vec{x}_1)}$

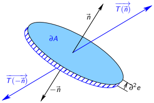
Cauchy stress tensor

On the pressure or tension in a solid body (1827)

Any part of a system in equilibrium is assumed to be in equilibrium itself, under the effect of the constrained vector applied to its boundary and a possible body force $\vec{f}$

Reciprocity of stress

- For a penny-shape of area ∂A, thickness ∂e and normal $\vec{n}$.
- $\partial^2 e \rightarrow 0 \Rightarrow$ the participation of stress exerted on the slice tends to 0.



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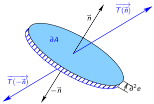

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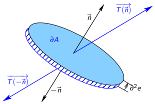

$$\overrightarrow{T(P,\vec{n})}\partial A + \overrightarrow{T(P,-\vec{n})}\partial A + \vec{f}\partial A \partial^2 e = \vec{0}$$

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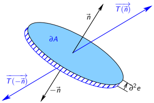

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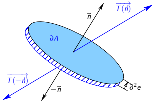

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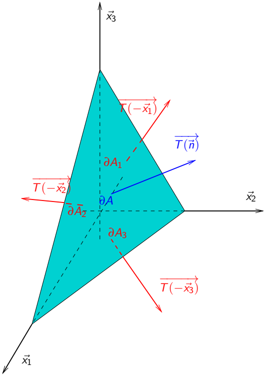
$$\overrightarrow{T(P,-n)} = -\overrightarrow{T(P,n)}$$

Linearity of the stress with respect to the normal vector :

- balance equations for a pyramidal volume around a given point P :
- $\overrightarrow{T(P,\vec{n})}\partial A + \overrightarrow{T(P,-\vec{x_1})}\partial A_1 + \overrightarrow{T(P,-\vec{x_2})}\partial A_2 + \overrightarrow{T(P,-\vec{x_3})}\partial A_3 = \vec{0}$
- $\vec{n} = \alpha\vec{x_1} + \beta\vec{x_2} + \gamma\vec{x_3}$, we have : $\partial A_1 = \alpha\partial A$, $\partial A_2 = \beta\partial A$, $\partial A_3 = \gamma\partial A$
- We then obtain : $\partial A\overrightarrow{T(P,\vec{n})} + \alpha\partial A\overrightarrow{T(P,-\vec{x_1})} + \beta\partial A\overrightarrow{T(P,-\vec{x_2})} + \gamma\partial A\overrightarrow{T(P,-\vec{x_3})} = \vec{0}$



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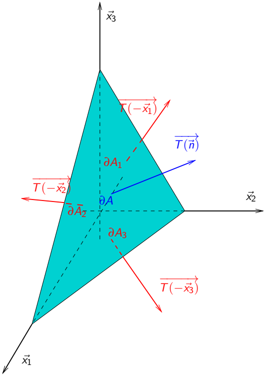

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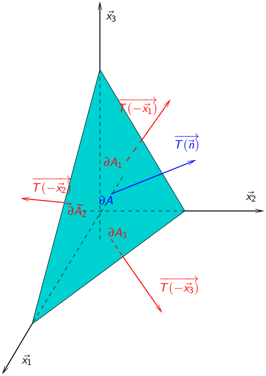

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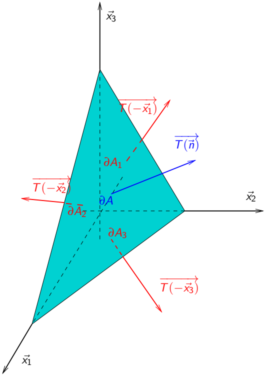

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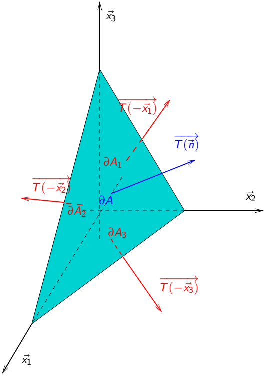

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Sress tensor

For $\overline{\overline{\sigma_P}}$, the function given at the point P which at a normal vector $\overrightarrow{n}$ associates the stress vector $\overrightarrow{T(P,\overrightarrow{n})}$, we have demonstrated that it is a linear form of the space $\mathbb{R}^3$ which allows it to be represented by a matrix.

$\overrightarrow{T(P,\overrightarrow{n})} = \overline{\overline{\sigma_P}}(\overrightarrow{n})$

$\overline{\overline{\sigma_P}}$ can be written with respect to the basis $(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3})$:

$$\overline{\overline{\sigma_P}} : \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}_{(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3})}$$

Symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$

Balance equations of a cubic elementary volume Ω cubic (or square in 2D) of infinitesimal dimensions ∂a. This element is loaded by the stress vectors exerted on each face (or edge).

- For the face of normal $\vec{x_1}$:
  $T(P, \vec{x_1}) = \sigma_{11}\vec{x_1} + \sigma_{21}\vec{x_2}$
- For the face of normal $\vec{x_2}$:
  $T(P, x_2) = \sigma_{12}\vec{x_1} + \sigma_{22}\vec{x_2}$
- For the face of normal$-\vec{x_1}$:
  $T(P, -\vec{x_1}) = -\sigma_{11}\vec{x_1} - \sigma_{21}\vec{x_2}$
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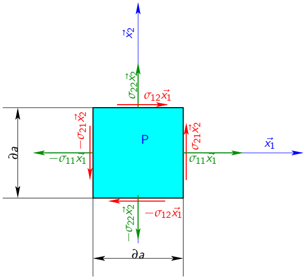

momentum equation around $P\vec{x_3}$

$$\frac{\partial a}{2}\vec{x_1} \wedge \sigma_{21}\partial a\vec{x_2} + \frac{\partial a}{2}\vec{x_2} \wedge \sigma_{12}\partial a\vec{x_1} - \frac{\partial a}{2}\vec{x_1} \wedge -\sigma_{21}\partial a\vec{x_2} - \frac{\partial a}{2}\vec{x_2} \wedge -\sigma_{12}\partial a\vec{x_1} = 0$$

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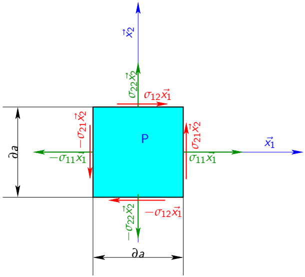

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$$\frac{\partial a}{2}\vec{x_1} \wedge \sigma_{21}\partial a\vec{x_2} + \frac{\partial a}{2}\vec{x_2} \wedge \sigma_{12}\partial a\vec{x_1} - \frac{\partial a}{2}\vec{x_1} \wedge -\sigma_{21}\partial a\vec{x_2} - \frac{\partial a}{2}\vec{x_2} \wedge -\sigma_{12}\partial a\vec{x_1} = 0$$

$$\frac{\partial a^2}{2}\sigma_{21} - \frac{\partial a^2}{2}\sigma_{12} + \frac{\partial a^2}{2}\sigma_{21} - \frac{\partial a^2}{2}\sigma_{12} = 0$$

Symmetry of the stress tensor $\sigma_{ij} = \sigma_{ji}$

the matrix $\overline{\overline{\sigma_P}}$ is symetric

$$\sigma_{12} = \sigma_{21}$$

$$\sigma_{ij} = \sigma_{ji}$$

the matrix $\overline{\overline{\sigma_P}}$ is diagonalisable

The eigenvalues of $\overline{\overline{\sigma_P}}$ are also called principal stresses noted σ_1, σ_2, σ_3.

$$\sigma_1 \geq \sigma_2 \geq \sigma_3$$

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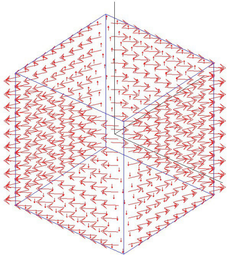
Balance equations :

- $\iint_{\partial\Omega} \overrightarrow{T(P,\vec{n})} dS + \iiint_{\Omega} \vec{f} dV = \vec{0}$
- $\iint_{\partial\Omega} \overline{\overline{\sigma_P}} \, \vec{n} \, dS + \iiint_{\Omega} \vec{f} dV = \vec{0}$
- $\oiint_{\partial\Omega} \overline{\overline{\sigma_P}} \overrightarrow{dS} + \iiint_{\Omega} \vec{f} dV = \vec{0}$



$$\overline{\overline{\sigma_P}} : \begin{bmatrix} 0.5 & 1 & 0.2 \\ 1 & 0.7 & 0 \\ 0.2 & 0 & 0.3 \end{bmatrix}$$

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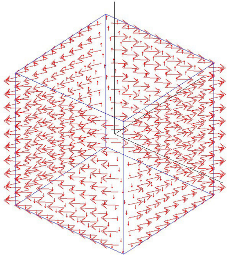

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- $\iint_{\partial\Omega} \overrightarrow{T(P,\vec{n})} dS + \iiint_{\Omega} \vec{f} dV = \vec{0}$
- $\iint_{\partial\Omega} \overline{\overline{\sigma_P}}\, \vec{n}\, dS + \iiint_{\Omega} \vec{f} dV = \vec{0}$
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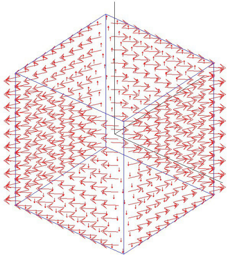

$$\overline{\overline{\sigma_P}} : \begin{bmatrix} 0.5 & 1 & 0.2 \\ 1 & 0.7 & 0 \\ 0.2 & 0 & 0.3 \end{bmatrix}$$

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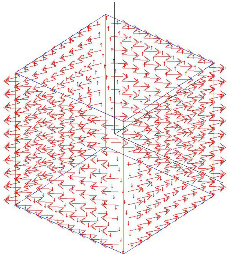

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Divergence theorem :

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Which demonstrates :

$$\overrightarrow{div \overline{\overline{\sigma_P}}} + \vec{f} = \vec{0}$$

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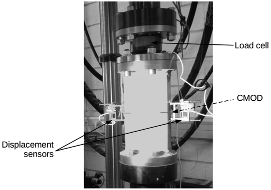
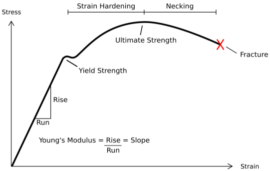
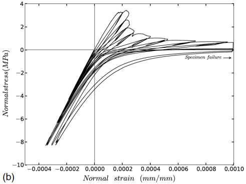
Uniaxial experiments

(a) For ductile material



(b) For quasi-brittle material (concrete)





Elastic stress strain relationship

Most materials exhibits a range where the relationship between stress and strain is reversible and linear.

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \; ; \; \sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

or

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \; ; \; \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Where E is the Young's modulus or modulus of elasticity and ν is the Poisson's ratio, S_{ijkl}, the matrix of compliance, C_{ijkl} its inverse, the matrix of elasticity.

Free energy

Energy of elasticity w_e

Is the quantity of energy stored into the material by deformation and by unit of volume. $w_e = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

Helmholtz free energy Ψ_e

is the elastic energy by mass unit, $\Psi_e = \frac{w_e}{\rho}$

$$\rho\Psi_e = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

The Helmholtz free energy is a state potential in terms of thermodynamics

$$\sigma_{ij} = \frac{\partial\rho\Psi}{\partial\varepsilon_{ij}} = C_{ijkl}\varepsilon_{ij}$$

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Experiments

Uniaxial

- Tension test on ductile material
- Compression test for concrete
- Splitting test (not really uniaxial)
- Tension test on concrete

multiaxial

- Tension torsion on a monocristal
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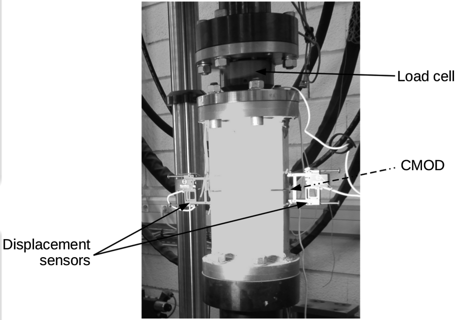

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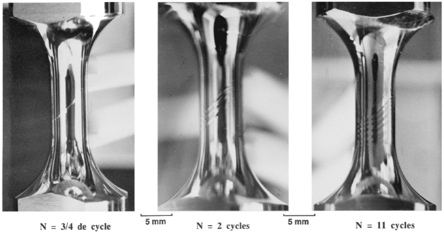

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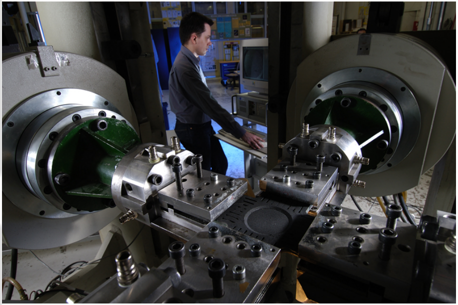

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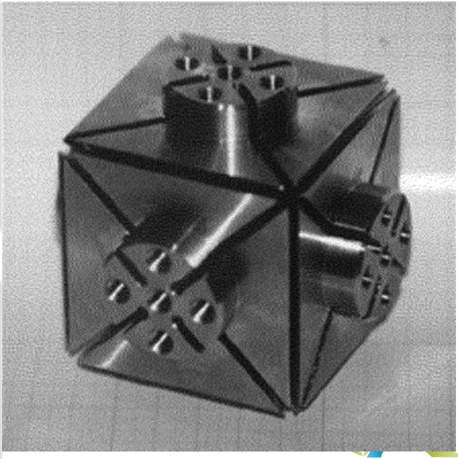

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How results of experiments can be used in a general case ?

results of experiments

- behaviour of a material in a particular condition
- mostly uniaxial
  - tension strength f_t
  - compression strength f_c

design a structure for a given loading

- The material must remain elastic in most cases, or its plasticity-damage must be acceptable
- Each point of the structure owns a particular state of stress
- The state of stress is often multi-axial

Problem ? ? ?

How to compare a tensor with a number ?

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The stress tensor and its associated matrix

"Stress tensor" and "Stress matrix"

- The stress tensor is the linear form
- $\overline{\overline{\sigma_P}} : \vec{n} \rightarrow \overrightarrow{T(P,\vec{n})}$
- The associated matrix of $\overline{\overline{\sigma_P}}$ for a given reference $(\vec{x}_1, \vec{x}_2, \vec{x}_3)$ is $[\sigma_{ij}]$
- The individual value of each σ_{ij} depends on the chosen reference and isn't physical representative.

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Invariants

Numbers calculated from matrix that independent of the reference

- $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = Tr(\sigma) = \sigma_{kk}$
- $I_2 = \frac{1}{2}\left((Tr\sigma)^2 - Tr\left(\sigma^2\right) \right)$
  $I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$
- $I_3 = Det(\sigma_{ij})$

In term of principal stresses

- $I_1 = \sigma_1 + \sigma_2 + \sigma_3$
- $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$
- $I_3 = \sigma_1\sigma_2\sigma_3$

Stress deviator :

Experiment results

- For a lot of materials, the elasticity domain is independent of the hydrostatic pressure $\pi = \frac{I_1}{3}$.
- The stress deviator is obtained by subtracting the hydrostatic pressure to the stress tensor.

$S_{ij} = \sigma_{ij} - \pi\delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$ is the Kronecker delta

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S32 & S_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S32 & S_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} - \pi & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \pi & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \pi \end{bmatrix}$$

Von-Mises equivalent stress (1913) :

invariants of the stress deviator tensor

The first, second and third invariants of the stress deviator tensor are called J_1, J_2 and J_3.

- $J_1 = Tr(S) = \sigma_{11} + \sigma_{22} + \sigma_{33} - 3\pi = 0$
- $J_2 = \frac{1}{2} Tr(S^2)$ (the sign convention is at the opposite of the definition given for I_2)
  $J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$
  $J_2 = \frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)$

Von Mises equivalent stress based on J_2

- must be homogeneous to a stress
- must be to the stress value in tension-compression

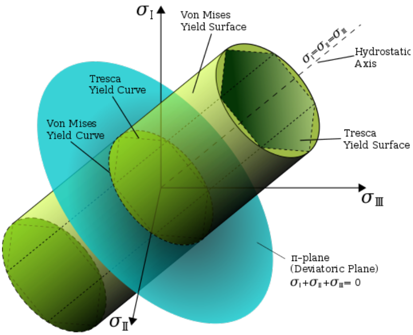
$$\sigma_{eq} = \sqrt{3J_2}$$

Von Mises Criterium of elasticity

The criterium of elasticity based on this equivalent stress is given by the equation :

$$\sigma_{eq} - \sigma_y = 0$$

Where σ_y is the yield stress in uniaxial stress condition.
The Von Mises stress is also known as the maximum energy of strain distorsion.



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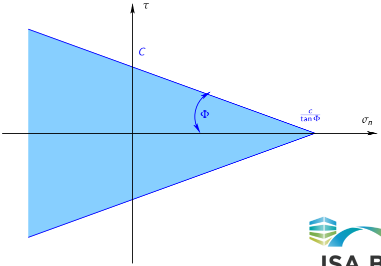
The Mohr Coulomb criterion

- Geomaterials (rocks, concrete) and most of quasi-brittle materials behaviour depends on the hydrostatic stress (ie different in tension and compression)
- The Coulomb criterium (1773), further studied by Mohr is based on the friction hypothesis

$$\tau = \sigma_n \tan \Phi + C$$

C : Cohesion

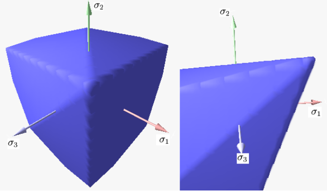
Φ : Friction angle



The Mohr Coulomb criterion

In terms of principal stresses $\sigma_1 \geq \sigma_2 \geq \sigma_3$

- $\sigma_1 - \sigma_3 = (\sigma_1 + \sigma_3)\sin\Phi + 2C\cos\Phi$
- The uniaxial tensile strength σ_t is given for $\sigma_t = \sigma_1$, $\sigma_2 = \sigma_3 = 0$:
  $\sigma_t = \frac{2C\cos\Phi}{1+\sin\Phi}$
- The uniaxial compressive strength σ_c is given for $\sigma_c = \sigma_3$, $\sigma_1 = \sigma_2 = 0$:
  $\sigma_c = -\frac{2C\cos\Phi}{1-\sin\Phi}$



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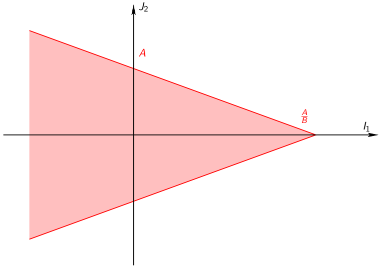
The Drucker Prager criterion (1952) :

- $$\sqrt{J_2} = A + BI_1$$
- In terms of σ_t and $sigma_c$
  - $A = \frac{2}{\sqrt{3}} \frac{\sigma_t \sigma_c}{\sigma_c - \sigma_t}$
  - $B = \frac{1}{\sqrt{3}} \frac{\sigma_t + \sigma_c}{\sigma_t - \sigma_c}$
- equivalence with Mohr-Coulomb Parameters
  - $A = \frac{6C \cos\Phi}{\sqrt{3}(3 - \sin\Phi)}$
  - $B = \frac{2 \sin\Phi}{\sqrt{3}(3 - \sin\Phi)}$



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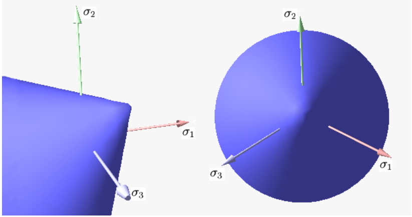


The Mazars' criterion (1984).

The Mazars' criterion is often use for concrete like materials modelled with damage. The idea of Mazars is that the non linearity of the material is generated by extension strains (i.e. positive strains).

$$\tilde{\varepsilon} = \sqrt{\langle\varepsilon_1\rangle^2 + \langle\varepsilon_2\rangle^2 + \langle\varepsilon_3\rangle^2}$$

where ε_1, ε_2 and ε_3 are the principal strains and $\langle\rangle$ designs the Macauley brackets $\langle x\rangle = \begin{cases} x & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$. The threshold function is therefore writen as following

$$\tilde{\varepsilon} - k(D) = 0$$

Where $k(D)$ is the yield strain depending of damage D. The inititial value is called ε_{d0} correspond to the limit of elasticity in positive strain.

The Mazars criterion (1984).

Limits of elasticity In the case of an uniaxial loading :

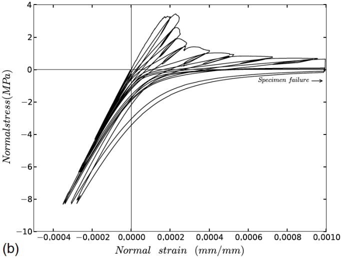
$$\overline{\overline{\sigma}} : \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ then } \overline{\overline{\varepsilon}} : \begin{bmatrix} \frac{\sigma_{11}}{E} & 0 & 0 \\ 0 & \frac{-\nu\sigma_{11}}{E} & 0 \\ 0 & 0 & \frac{-\nu\sigma_{11}}{E} \end{bmatrix}$$

in tension

$\sigma_{11} = \sigma_t > 0 \Rightarrow \varepsilon_{11} = \frac{\sigma_t}{E} > 0 \,;\, \varepsilon_{22} < 0 \,;\, \varepsilon_{33} < 0$

$\tilde{\varepsilon} = \frac{\sigma_t}{E} = \varepsilon_{d0}$

$$\sigma_t = E\varepsilon_{d0}$$



(b)

The Mazars criterion (1984).

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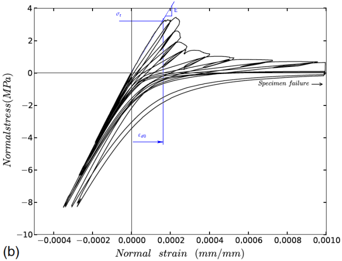

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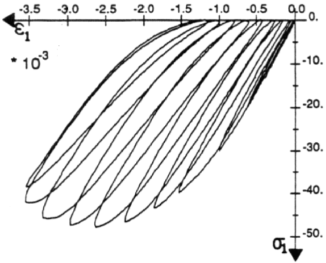
in compression

$\sigma_{11} = \sigma_c < 0 \Rightarrow \varepsilon_{11} = \frac{\sigma_t}{E} > 0$; $\varepsilon_{22} < 0$; $\varepsilon_{33} < 0$
$\tilde{\varepsilon} = -\sqrt{2}\nu\frac{\sigma_c}{E} = \varepsilon_{d0}$

$$\sigma_c = -\frac{E\varepsilon_{d0}}{\sqrt{2}\nu}$$



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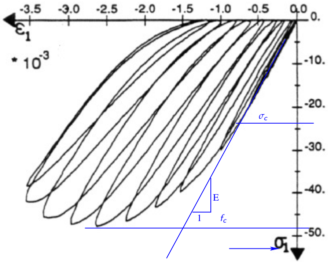

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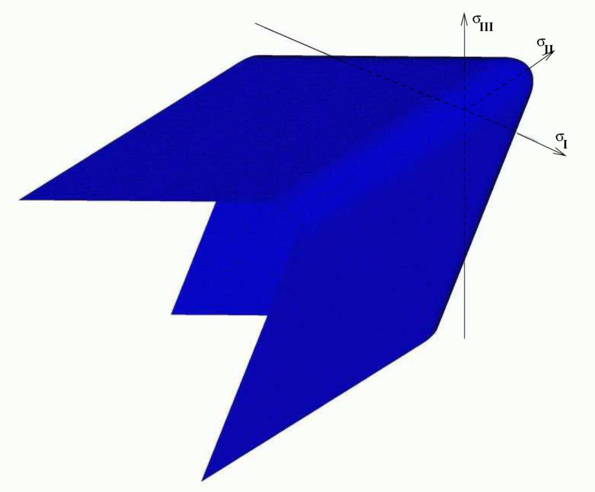
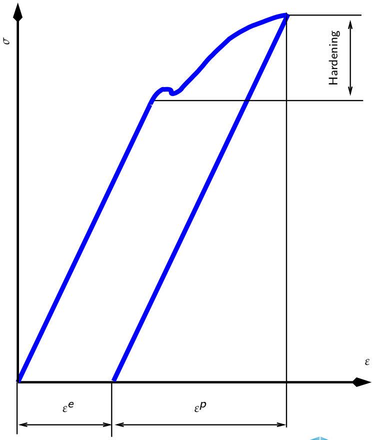


plasticity

- Plasticity is directly related to local slips / dislocations [1]
- $\varepsilon = \varepsilon^e + \varepsilon^p$
- elastic slope remains constant
- hardening or softening

$$\sigma = E\varepsilon^e = E\left(\varepsilon - \varepsilon^p\right)$$



Plastic strains

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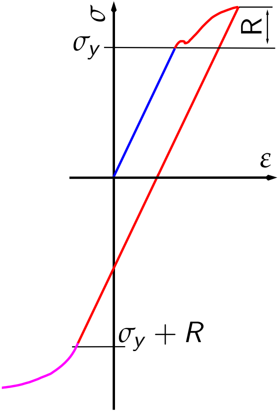
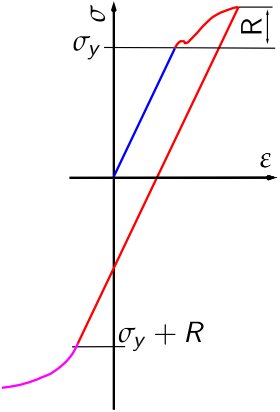
Isotropic Hardening



Isotropic hardening

Isotropic Hardening



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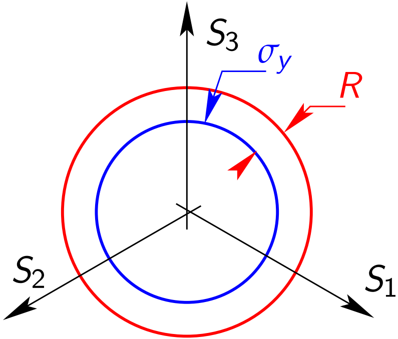
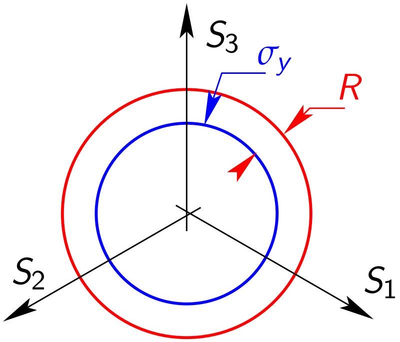


Isotropic yield function

Isotropic Hardening

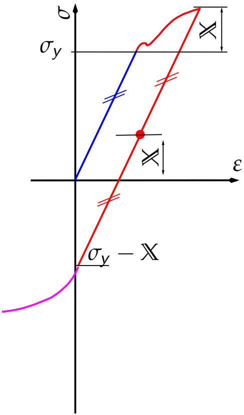
yield function

- The limit of elasticity is increased of the hardening value R
- $\sigma_{eq} - \sigma_y - R = 0$
- R is the hardening parameter (scalar)



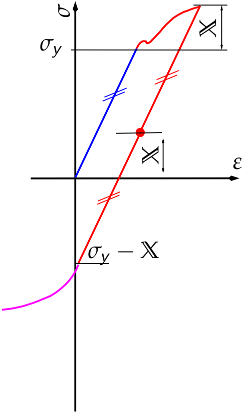
Isotropic yield function

kinematic Hardening

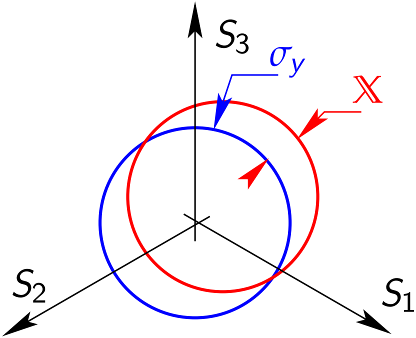


IKinematic hardening

kinematic Hardening



IKinematic hardening

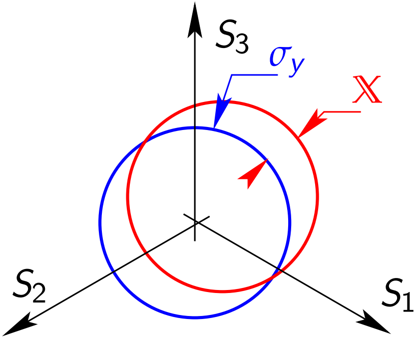


Kinematic yield function

kinematic Hardening

yield function

- The "center of elasticity" is moved by the hardening value $\mathbb{X}$
- $\sigma_{eq}\left(\sigma - \mathbb{X}\right) - \sigma_y = 0$
- $\mathbb{X}$ is the hardening parameter (tensor)

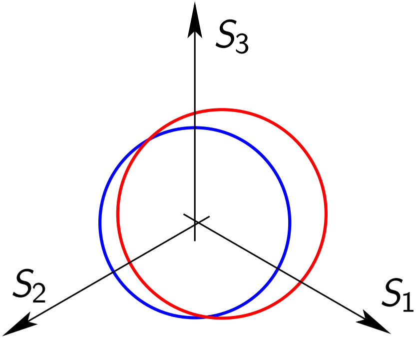


Kinematic yield function

Combined hardening

yield function

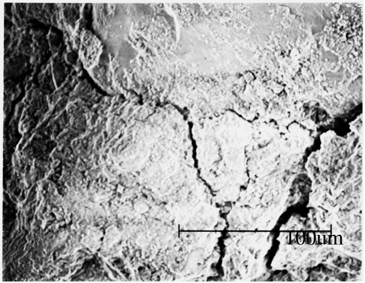
- The general cases combines Kinematic and Isotropic hardenings
- $\sigma_{eq}\left(\sigma - \mathbb{X}\right) - \sigma_y - R = 0$



Kinematic yield function

Damage

The damage is linked to debonding of material and microcracking that accours at the mesoscopic level [1]

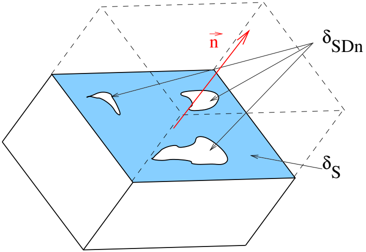


Microcracking into concrete

Damage

- Let δS be the intersection area of a given plane of normal $\vec{n}$ with a Representative Elementary Volume (RVE).
- Let δS_{Dn} be the effective area of micro-cracks and micro-cavities within the intersection plane at the point M
- The value of damage is then defined by

$$D(M, \vec{n}) = \frac{\delta S_{Dn}}{\delta S}$$



projection of defects on a plane after Kachanov

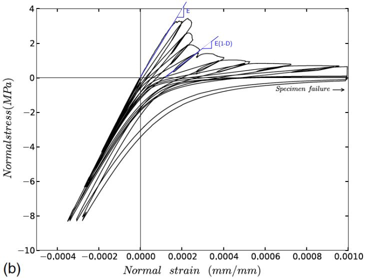
Isotropic Damage

Classical hypothesis

- Damage is isotropic
- The damage variable is a real

$0 < D < 1$

- $D = 0 \longrightarrow$ undamaged material
- $D = 1 \longrightarrow$ fully broken material
- $\sigma = E\,(1 - D)\,\varepsilon^e$ (1D)

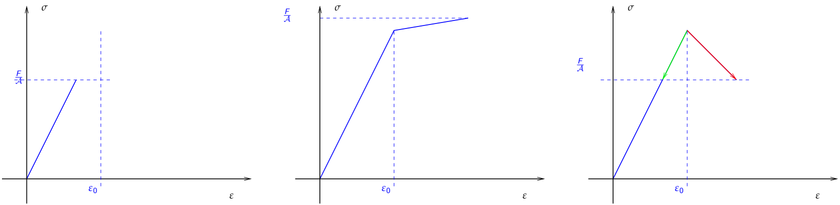


Stress-strain relation exhibiting damage

Hardening and Softening

Non linear behavior during a 1D tension test

- Stress is increasing $\Longrightarrow$ hardening
- Stress is decreasing $\Longrightarrow$ softening.



(a) Elastic (b) Hardening (c) Softening

Problem

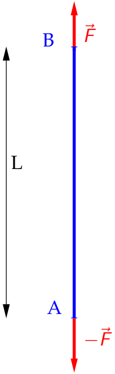
2 solutions for a given load in case of softening

1D tension test

tension test of a simple 1D bar AB

Test

- Initial length L
- Section Area $\mathcal{A}$
- Test performed by increasing the elongation δL



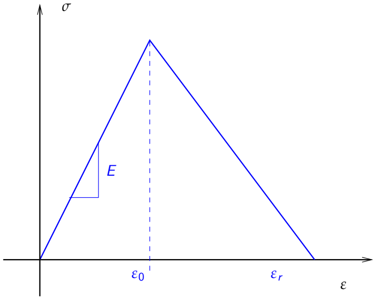
tension test of a simple 1D bar AB

Test

- Initial length L
- Section Area $\mathcal{A}$
- Test performed by increasing the elongation δL

Material behaviour

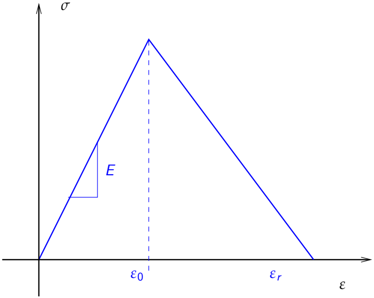
- E is the modulous of elasticity
- ε_0 is the threshold in tension
- ε_r is the fracture strain



tension test of a simple 1D bar AB

Stress strain relationship

$$\begin{cases} \varepsilon < \varepsilon_0 & \sigma = E\varepsilon \\ \varepsilon > \varepsilon_0 & \sigma = \frac{E\varepsilon_0}{\varepsilon_0 - \varepsilon_r}\left(\varepsilon - \varepsilon_r\right) \end{cases}$$



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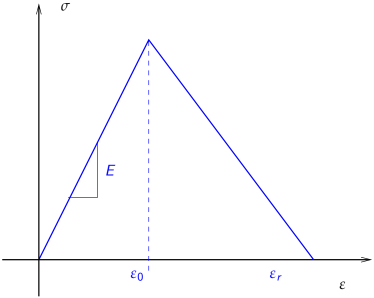
Balance equations

- $\frac{\partial \sigma}{\partial x} = 0 \Rightarrow \sigma = \frac{F}{\mathcal{A}}$
- The stress is homogeneous along the bar
- 2 solutions : elastic path or non linear path



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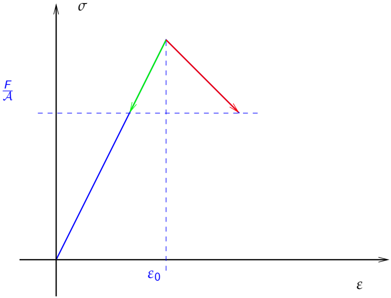

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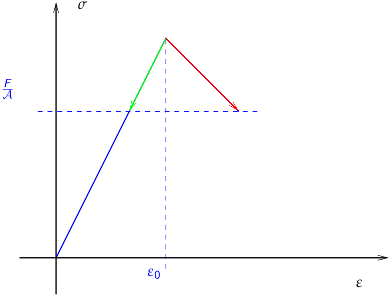
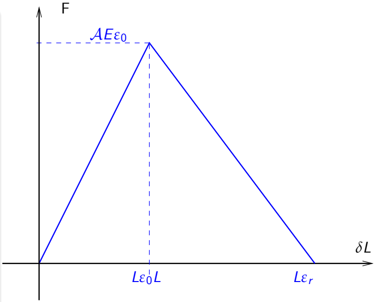


At least one point must follow the non linear path

Homogeneous solution

Hypothesis

- All points are following the non linear path
- Plastic strains or damage are identical along the bar
- Total strains are identical along the bar
- The final length of the bar is $L\varepsilon_r$



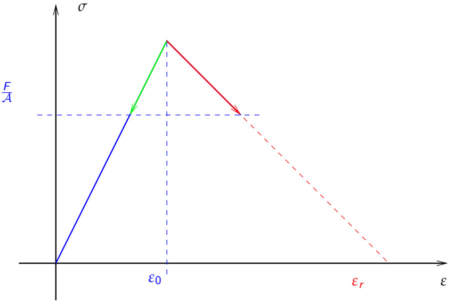
Etherogeneous solution

Hypothesis

- a part of the specimen is following the non linear path while the other is following the linear path.

$\sigma = \frac{F}{\mathcal{A}}$

- $\varepsilon = \frac{\sigma}{E} = \frac{F}{E\mathcal{A}}$
- $\varepsilon = \dots$



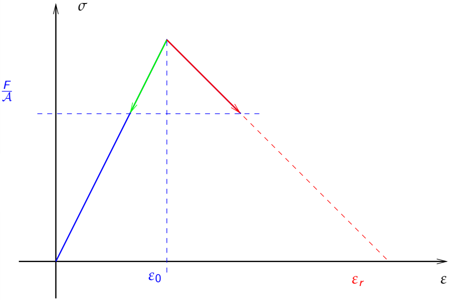
Etherogeneous solution

Hypothesis

- a part of the specimen is following the non linear path while the other is following the linear path.

$\sigma = \frac{F}{\mathcal{A}}$

- $\varepsilon = \frac{\sigma}{E} = \frac{F}{E\mathcal{A}}$
- $\varepsilon = \varepsilon_r - \frac{F}{E\mathcal{A}} \frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}$



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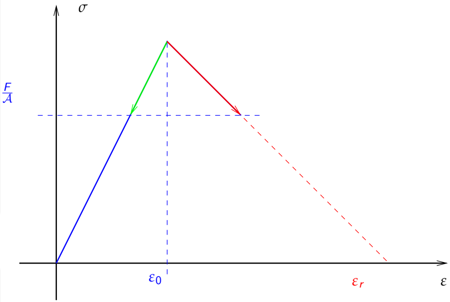
Etherogeneous solution

Hypothesis

- The length of the non linear part is $\frac{L}{n}$,
- The lengh of the linear part is $L\frac{n-1}{n}$
- $n \in \mathbb{N}, n > 1.$

Elongation

- $\delta L_e = \ldots$
- $\delta L_{nl} = \ldots$



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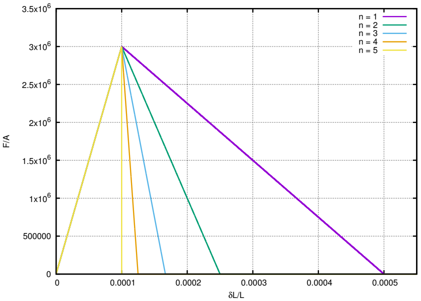
Elongation

- $\delta L_e = \frac{F}{E\mathcal{A}} L \frac{n-1}{n}$
- $\delta L_{nl} = \left(\varepsilon_r - \frac{F}{E\mathcal{A}} \frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}\right) \frac{L}{n}$



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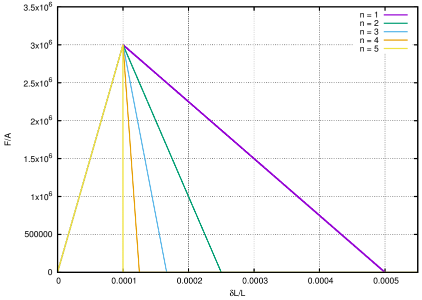

$$\delta L = \frac{F}{E\mathcal{A}} L \frac{n-1}{n} + \left(\varepsilon_r - \frac{F}{E\mathcal{A}} \frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}\right) \frac{L}{n}$$

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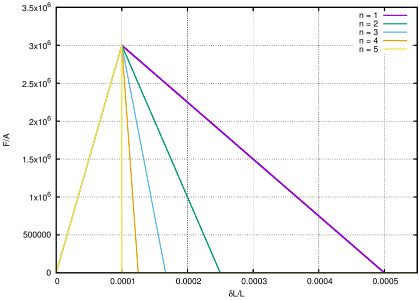
Etherogeneous solution

Breaking elongation δL_r

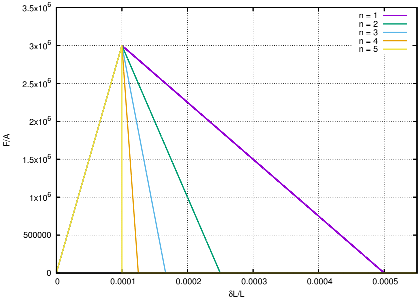
- $F = 0$
- $\delta L_r = \varepsilon_r \frac{L}{n}$
- strain of elastic elements $= 0$.
- strain of non linear elements ε_r
- $\frac{\varepsilon_c r}{n} < \varepsilon_0 \implies \delta L_r < \delta L_e = L\varepsilon_0 \implies$ instability



Etherogeneous solution

Dissipated energy W_f

- $W_f = \int_0^{\delta L_r} F \, \mathrm{d}\delta L$
- W_f is the area under the curve
- $W_f = \frac{E \varepsilon_0 \mathcal{A}}{2} \varepsilon_r \frac{L}{n}$
- $\lim_{n \to +\infty} W_f = 0$



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