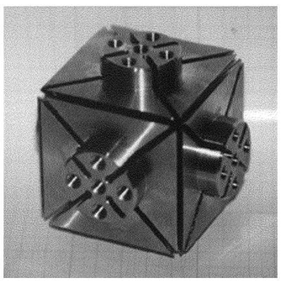

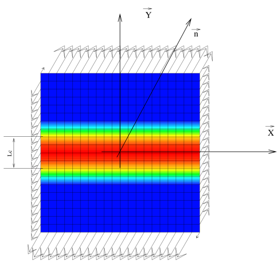

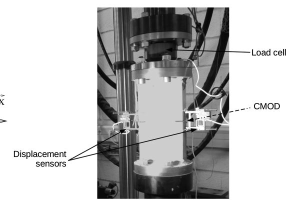

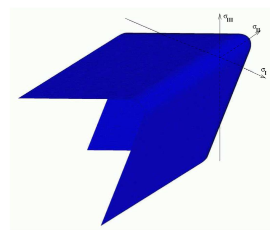

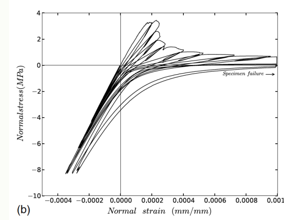

Advanced mechanics

Non linear behaviour of materials

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ISABTP 5 / Master PSCE

Year 2023-2024

Part I
Reminders on elasticity of materials

1 stress :

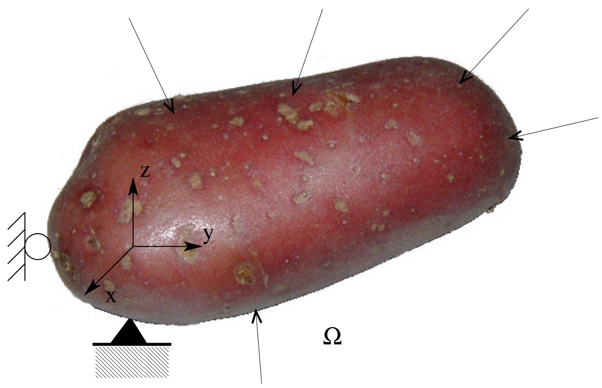
Stresses represent the cohesive forces in a solid that allow the material to withstand the loading.

Stresses are the result of interaction between small parts of the material (crystals, molecules... etc... etc...).

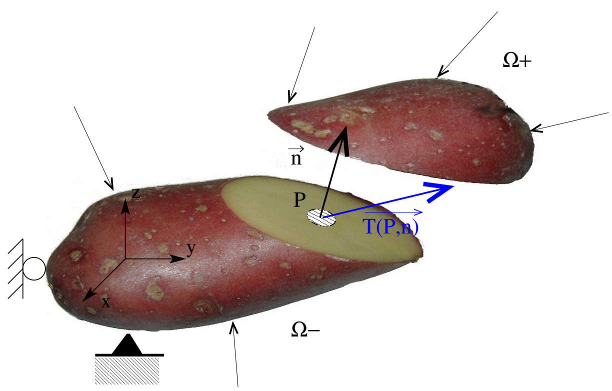
The equivalent of stress for a perfect fluid is pressure.

1.1 Definition of the stress vector :

For a solid Ω loaded by a set of mechanical actions and in balance with respect to a reference system, the balance equations are verified for an arbitrary part of Ω.



If we cut Ω by a plane of normal $\overrightarrow{n}$ passing through point P, the two parts Ω^+ located on the normal side and Ω^- located on the opposite side, are in balance.



Ω^- is in balance under the effect:

- External forces exerted on it.
- From the stress vector $\overrightarrow{T(P,\overrightarrow{n})}$ exerted at any point P of the cut-off plane.

The stress vector $\overrightarrow{T(P,\overrightarrow{n})}$ is the surface density of the forces exerted by Ω^+ on Ω^-. It is the physical variable associated with the stress.

Remarks:

1. The stress vector is homogeneous with an effort per unit area or pressure, it is expressed in Pascals.
   $1Pa = 1N/m^2,\ 1MPa = 10^6Pa,\ 1kPa = 10^3Pa,\ 1GPa = 10^9Pa$

There are also more exotic units:
$1T/m^2 \simeq 10kPa,\ 1kg/cm^2 \simeq 100kPa,\ 1bar = 100kPa,\ 1PSI \simeq 6,9MPa$
The use of these units is strongly discouraged.

2. If at a point P we perform two normal cut-off planes $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$, we obtain two constrained vectors $T\left(P,\overrightarrow{n_1}\right)$ and which are $T\left(P,\overrightarrow{n_2}\right)$ a priori different.

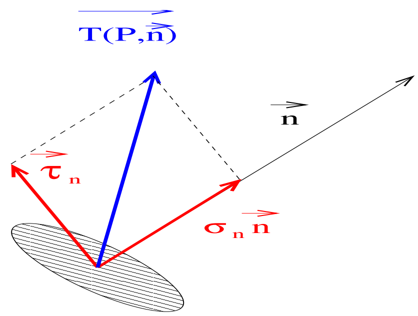
3. At two points P and Q of the same normal cut-off plane $\overrightarrow{n}$ we obtain two stress vectors $T\left(P,\overrightarrow{n}\right)$ and $T\left(Q,\overrightarrow{n}\right)$ which are different.

4. The torsor of actions resulting from Ω^+'s actions on Ω^- for a normal cut-off plane Π of normal $\overrightarrow{n}$ is :
$$\left\{ \begin{array}{c} \overrightarrow{F_{\Omega^+/\Omega^-}} = \int\int_\Pi \overrightarrow{T(P,\overrightarrow{n})}ds \\ \overrightarrow{M_{A,\Omega^+/\Omega^-}} = \int\int_\Pi \overrightarrow{AP} \wedge \overrightarrow{T(P,\overrightarrow{n})}ds \end{array} \right\}_A$$

1.2 Projections of the stress vector :

1.2.1 Norman and tangential stresses:

The stress vector is composed of a normal stress σ_n and a tangential stress $\overrightarrow{\tau_n}$.



$$\sigma_n = \overrightarrow{T(P,\overrightarrow{n})} \bullet \overrightarrow{n} \qquad \overrightarrow{\tau_n} = \overrightarrow{T(P,\overrightarrow{n})} \bullet \overrightarrow{n} - \sigma_n \overrightarrow{n}$$

- The normal stress σ_n is a number.
- The tangential stress $\overrightarrow{\tau_n}$ is a vector.

1.2.2 Projections on the reference vectors:

For a given reference system $(\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3})$ of orthonormal vectors, the projections of the stress vector on the basis vector are called as following:

$$\sigma_{ij} = \overrightarrow{T(P,\overrightarrow{x_i})} \bullet \overrightarrow{x_j}$$

1.3 Cauchy stress tensor [1]:

Any part of a system in equilibrium is assumed to be in equilibrium itself, under the effect of the constrained vector applied to its boundary and a possible body force $\vec{f}$

1.3.1 Reciprocity of stress

For an infinitesimal part around a given point for which the thickness is small with respect to other dimensions, we can take a penny-shape of area ∂A, thickness $\partial^2 e$ and normal $\vec{n}$ (figure 2).

If the thickness $\partial^2 e$ tends towards 0, the participation of stress exerted on the slice is neglectible and the balance equations reads:
$\overrightarrow{T(P,\overrightarrow{n})}\partial A + \overrightarrow{T(P,-\overrightarrow{n})}\partial A + \vec{f}\partial A\partial e = \vec{0}$, since $\partial e \ll 1$ the body forces can be neglected and we can write :

[1] Cauchy, De la pression ou tension dans un corps solide, [On the pressure or tension in a solid body], Exercices de Mathématiques, vol. 2, p. 42 (1827)

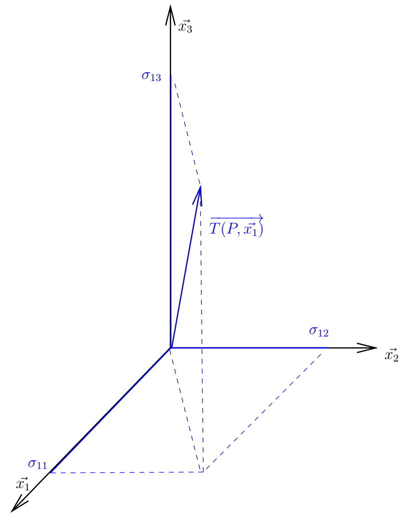


Figure 1: Projections of $\overrightarrow{T(P,\vec{x_1})}$

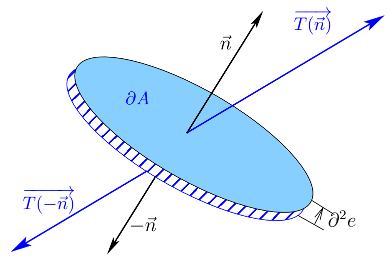


Figure 2: reciprocity of the stress vector

$$\boxed{\overrightarrow{T(P,-\vec{n})} = -\overrightarrow{T(P,\vec{n})}}$$

1.3.2 Linearity of the stress with respect to the normal vector :

For a pyramidal volume element of infinitesimal dimensions around a given point P (figure 3).

$\overline{\text{The volume element is in balance}}$

$\overrightarrow{T(P,\vec{n})}\partial A + \overrightarrow{T(P,-\vec{x_1})}\partial A_1 + \overrightarrow{T(P,-\vec{x_2})}\partial A_2 + \overrightarrow{T(P,-\vec{x_3})}\partial A_3$

If the normal vector is $\vec{n} = \alpha\vec{x_1} + \beta\vec{x_2} + \gamma\vec{x_3}$, we have : $\partial A_1 = \alpha\partial A$, $\partial A_2 = \beta\partial A$, $\partial A_3 = \gamma\partial A$

We then obtain: $\partial A\overrightarrow{T(P,\alpha\vec{x_1} + \beta\vec{x_2} + \gamma\vec{x_3})} + \alpha\partial A\overrightarrow{T(P,-\vec{x_1})} + \beta\partial A\overrightarrow{T(P,-\vec{x_2})} + \gamma\partial A\overrightarrow{T(P,-\vec{x_3})} = 0$

which demonstrates that the stress vector expression is linear with respect to the normal vector :

$$\boxed{\overrightarrow{T(P,\alpha\vec{x} + \beta\vec{y} + \gamma\vec{z})} = \alpha\overrightarrow{T(P,\vec{x})} + \beta\overrightarrow{T(P,\vec{y})} + \gamma\overrightarrow{T(P,\vec{z})}}$$

1.3.3 Sress tensor

For $\overline{\overline{\sigma_P}}$, the function given at the point P which at a normal vector $\overrightarrow{n}$ associates the stress vector $\overrightarrow{T\left(P,\overrightarrow{n}\right)}$, we have demonstrated that it is a linear form of the space $\mathbb{R}^3$ which allows it to be represented by a matrix.

$\overrightarrow{T\left(P,\overrightarrow{n}\right)} = \overline{\overline{\sigma_P}}\left(\overrightarrow{n}\right)$

$\overline{\overline{\sigma_P}}$ can be written with respect to the basis $(\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3})$:

$$\overline{\overline{\sigma_P}} : \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}_{(\overrightarrow{x_1},\overrightarrow{x_2},\overrightarrow{x_3})}$$

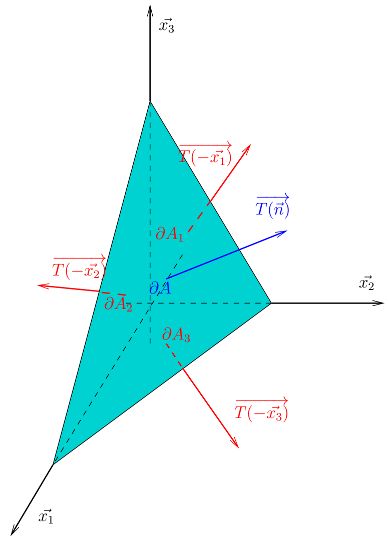


Figure 3: Evidence of the stress tensor

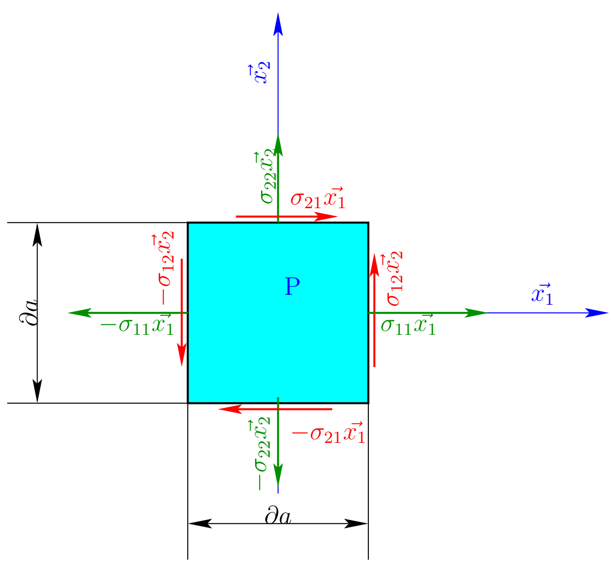


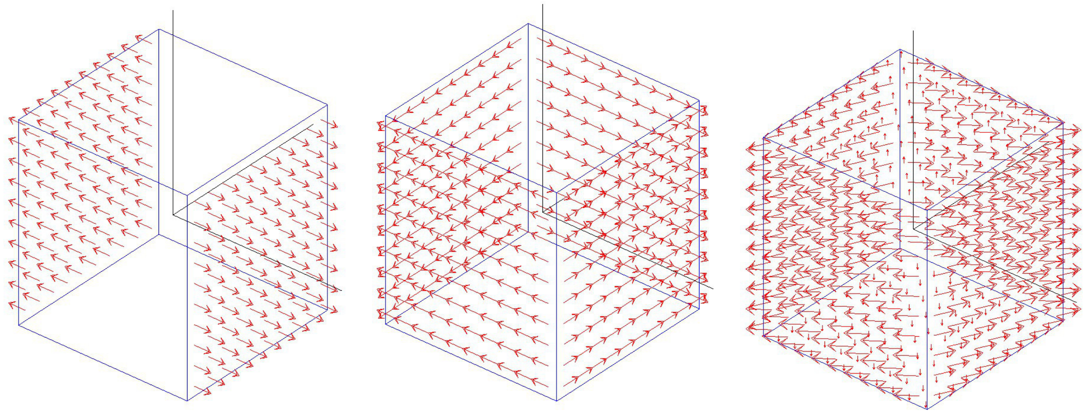
Figure 4: Symmetry of the stress tensor

1.3.4 Symmetry of the stress tensor

We can show that the stress tensor $\overline{\overline{\sigma_P}}$ is symmetric ($\sigma_{ij} = \sigma_{ji}$),let us take as an example a cubic elementary volume Ω cubic (or square in 2D) of infinitesimal dimensions ∂a. This element is loaded by the stress vectors exerted on each face (or edge).

- For the face of normal $\vec{x_1}$: $T(P, \vec{x_1}) = \sigma_{11}\vec{x_1} + \sigma_{12}\vec{x_2}$
- For the face of normal $\vec{x_2}$: $T(P, x_2) = \sigma_{21}\vec{x_1} + \sigma_{22}\vec{x_2}$
- For the face of normal$-\vec{x_1}$: $T(P, -\vec{x_1}) = -\sigma_{11}\vec{x_1} - \sigma_{12}\vec{x_2}$
- For the face of normal$-\vec{x_2}$: $T(P, -\vec{x_2}) = -\sigma_{21}\vec{x_1} - \sigma_{22}\vec{x_2}$

The momentum balance equation reads
$\frac{\partial a}{2}\vec{x_1} \wedge \sigma_{21}\partial a\vec{x_2} + \frac{\partial a}{2}\vec{x_2} \wedge \sigma_{12}\partial a\vec{x_1} - \frac{\partial a}{2}\vec{x_1} \wedge -\sigma_{21}\partial a\vec{x_2} - \frac{\partial a}{2}\vec{x_2} \wedge -\sigma_{12}\partial a\vec{x_1} = 0$
then :
$\frac{\partial a^2}{2}\sigma_{21} - \frac{\partial a^2}{2}\sigma_{12} + \frac{\partial a^2}{2}\sigma_{21} - \frac{\partial a^2}{2}\sigma_{12} = 0$



(a) $\overline{\overline{\sigma_P}} : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\overline{\overline{\sigma_P}} : \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (c) $\overline{\overline{\sigma_P}} : \begin{bmatrix} 0.5 & 1 & 0.2 \\ 1 & 0.7 & 0 \\ 0.2 & 0 & 0.3 \end{bmatrix}$

Figure 5: stress vector on an inner surface

$$\boxed{\sigma_{12} = \sigma_{21}}$$

Which shows that the matrix $\overline{\overline{\sigma_P}}$ is symetric, it is as a consequence diagonalizable. The eigenvalues of $\overline{\overline{\sigma_P}}$ are also called principal stresses noted σ_1, σ_2, σ_3.

1.4 Balance equations:

If we take an arbitrary elementary volume Ω a the neibourhood of the point P, one can say that Ω is in equilibrium under the action of the stresses vectors exerted on its boundary surface further called $\partial\Omega$ and under the action of possible body force $\vec{f}$. An example of drawing of the vector stress on a boundary of a cube is ploted on the figure 5. The sum of applied stresses and body forces must be zero in order to satisfy the balance equations.

We obtain as a consequence $\iint_{\partial\Omega} \overrightarrow{T(P,n)} dS + \iiint_\Omega \vec{f} dV = \vec{0}$ that can also be written $\iint_{\partial\Omega} \overline{\overline{\sigma_P}} \vec{n} \, dS + \iiint_\Omega \vec{f} dV = \vec{0}$ or

$$\oiint_{\partial\Omega} \overline{\overline{\sigma_P}} \overrightarrow{dS} + \iiint_\Omega \vec{f} dV = \vec{0}$$

The divergence theorem also known as Green-Ostrogradski theorem[2] states that the outward flux of a tensor field through a closed surface is equal to the volume integral of the divergence over the region inside the surface :

$$\oiint_{\partial\Omega} \overline{\overline{\sigma_P}} \overrightarrow{dS} = \iiint_\Omega \overrightarrow{div \overline{\overline{\sigma_P}}} dV$$

We have then

$$\iiint_\Omega \overrightarrow{div \overline{\overline{\sigma_P}}} + \vec{f} dV = \vec{0}, \quad \forall \Omega$$

Which demonstrates :

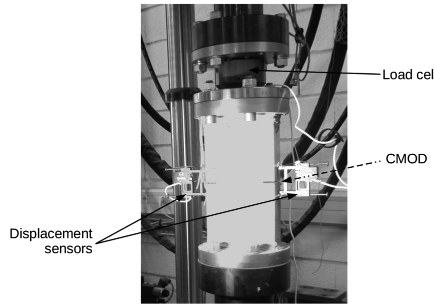
$$\boxed{\overrightarrow{div \overline{\overline{\sigma_P}}} + \vec{f} = \vec{0}}$$

2 Elastic stress strain relationship

Most materials exhibits a range where the relationship between stress ans strain is reversible, this domain is called elasticity domain. Within this elasticity domain, the behaviour of materials can be supposed in most case as linear (see Figure 7).

In the case of an isotropic material :

[2] Ostrogradski, «Proof of a theorem in Integral Calculus». paper presented at the "académie des sciences de Paris" on février 13, 1826



(a) For ductile material [2] (b) For quasi-brittle material (concrete) [9]

Figure 6: Tensile tests

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \ ; \ \sigma_{ij} = \frac{E}{1+\nu}\left(\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right)$$

or

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \ ; \ \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Where E is the Young's modulus or modulus of elasticity and ν is the Poisson's ratio, S_{ijkl}, the matrix of compliance, C_{ijkl} its inverse, the matrix of elasticity.

2.1 Free energy

We can define on this basis the energy of elasticity which is the quantity of energy stored into the material by deformation and by unit of volume.

$w_e = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$

The Helmholtz free energy Ψ_e is the elastic energy by mass unit, it is expressed in terms of strain variable (The equivalent Gibb's energy expressed in terms of stress is less used)

$$\rho\psi_e = \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$$

The Helmholtz free energy is a state potential in terms of thermodynamics, the state variable is the strain ε_{ij} and the stress σ_{ij} is the associated variable is obtained by derivation of the potential.

$$\sigma_{ij} = \frac{\partial\rho\Psi}{\partial\varepsilon_{ij}}$$

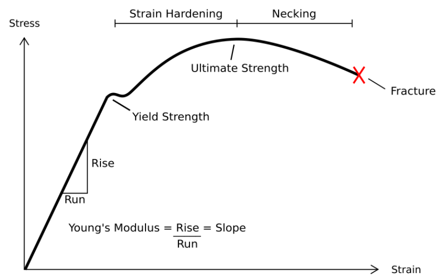
Part II
Criteria of elasticity

3 Uniaxial experiments :

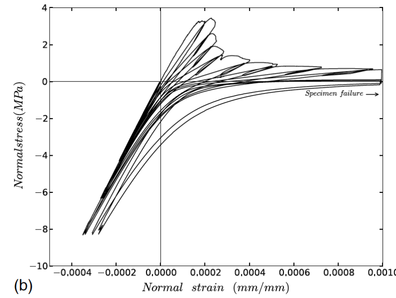
Materials are often caracterized from uniaxial experiments: i.e.

$$\overline{\overline{\sigma}} : \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

When used in a real structure, material are often loaded under multiaxial conditions and the stress (and the strain) are caracterized by full tensor, the idea of criteria of elasticity is to calculate a number based on the state of stress (or strain) which characterizes an equivalent state of the material if it would be loaded under uni-axial condition. In order to develop such kind of criteria, it is usefull to perform multi-axial experiments such as tension-torsion test, biaxial or triaxial tests.



(a) For ductile material



(b) For quasi-brittle material (concrete)[9]

Figure 7: Behaviour in tension

4 Invariants of the stress tensor :

The stress tensor components σ_{ij} with respect to a given basis are dependant on the basis and canoot be taken as physical variables. Nevertheless the tensor itself which is a linear form isn't dependant on the basis used to project the matrix.

Some numbers calculated from the matrix components are independant on the basis, they can be taken as physical parameters. Usefull examples are :

- $I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = Tr(\sigma) = \sigma_{kk}$
- $I_2 = \frac{1}{2}\left((Tr\sigma)^2 - Tr\left(\sigma^2\right)\right)$
  $I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2$
- $I_3 = Det(\sigma_{ij})$

The invariants can be written in terms of principal stresses :

- $I_1 = \sigma_1 + \sigma_2 + \sigma_3$
- $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3$
- $I_3 = \sigma_1\sigma_2\sigma_3$

4.1 Stress deviator :

It is observed that for a lot of materials, the elasticity domain is independent of the hydrostatic pressure $\pi = \frac{I_1}{3}$. We can build the so called stress deviator substracting the hydrostatic pressure to the stress tensor.

$S_{ij} = \sigma_{ij} - \pi\delta_{ij}$ where $\delta_{ij} = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$ is the Kronecker delta

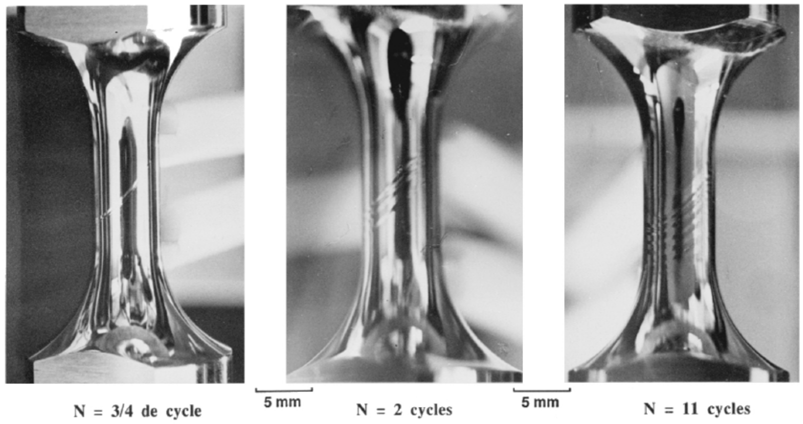
$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S32 & S_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} - \begin{bmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{bmatrix}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S32 & S_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{11} - \pi & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \pi & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \pi \end{bmatrix}$$

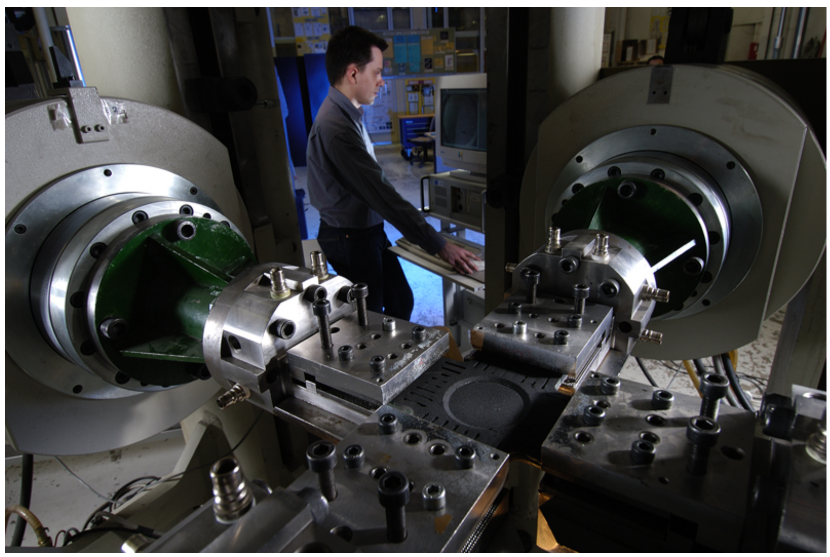
4.2 Invariants of the stress deviator tensor :

The first, second and third invariants of the stress deviator tensor are called J_1, J_2 and J_3.

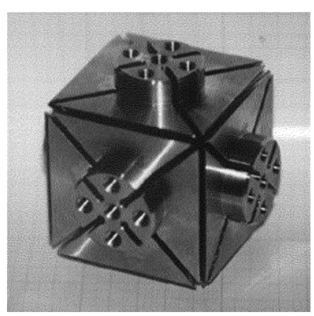
- $J_1 = Tr(S) = \sigma_{11} + \sigma_{22} + \sigma_{33} - 3\pi = 0$
- $J_2 = \frac{1}{2}Tr(S^2)$ (the sign convention is at the opposite of the definition given for I_2)
  $J_2 = \frac{1}{2}(S_1^2 + S_2^2 + S_3^2)$
  $J_2 = \frac{1}{6}\left((\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right)$



(a) Tension-torsion of a monochristal[5]



(b) Bi-traction[1]



(c) 3D tension-compression[6]

(d) Biaxial test on concrete

Figure 8: Multiaxial experiments

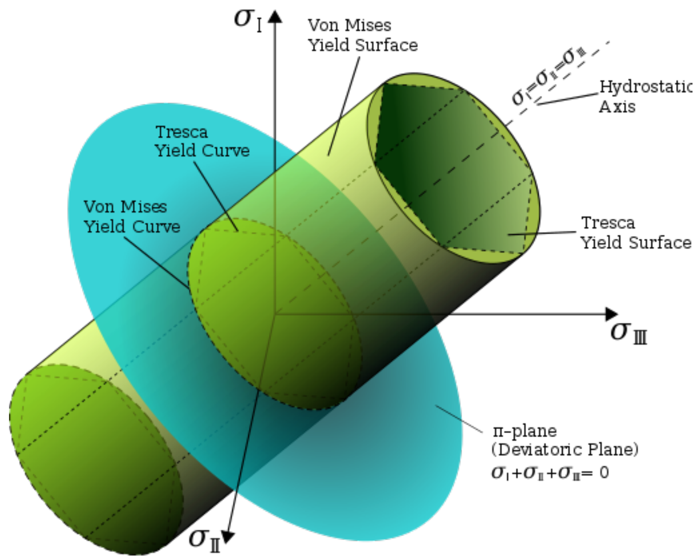


Figure 9: Von Mises criterium (Picture courtesy of Rswarbrick)

4.3 Von-Mises equivalent stress [10]:

The idea of Von Mises is to create an equivalent stress σ_{eq} based on the second invariant of the stress deviator. In order to find the applied stress in the uniaxial case, the expression of the equivalent stress is :

$$\sigma_{eq} = \sqrt{\frac{3}{2}J_2}$$

The criterium of elasticity based on this equivalent stress is given by the equation :

$$\sigma_{eq} - \sigma_y = 0$$

Where σ_y is the yield stress in uniaxial stress condition.

The Von Mises stress is also known as the maximum energy of strain distosion. The Von Mises criterium is presented in the space of principal stresses with respect to the Tresca criterium which is the maximum shear stress figure 9.

The

4.4 Criteria that accounts for the hydrostatic stress

Geomaterials (rocks, concrete) and most of quasi-brittle materials behaviour depends on the hysdrostatic stress the well known criteria are :

4.4.1 The Mohr Coulomb criteria :

The Coulomb criteriuù [3], further sudied by Mohr is based on the friction hypothesis. For a material of a defined cohesion C and a frition angle Φ, the limit state is given by the condition $\tau_y = \sigma tan\Phi + C$ where τ_y is the shear strength and σ is the normal stress.

This condition drives to the following equation :

$$\sigma_1 - \sigma_3 + (\sigma_1 + \sigma_3)\sin\Phi - 2C\cos\Phi = 0$$

Where $\sigma_1 \geq \sigma_2 \geq \sigma_3$

The drawing of the Mohr-Coulomb criterium in the principal stresses space is given in the figure 10:

The uniaxial tensile strength σ_t is given for $\sigma_t = \sigma_1$, $\sigma_2 = \sigma_3 = 0$: $\sigma_t = \frac{2C\cos\Phi}{1+\sin\Phi}$

The uniaxial compressive strength σ_c is given for $\sigma_c = \sigma_3$, $\sigma_1 = \sigma_2 = 0$: $\sigma_c = -\frac{2C\cos\Phi}{1-\sin\Phi}$

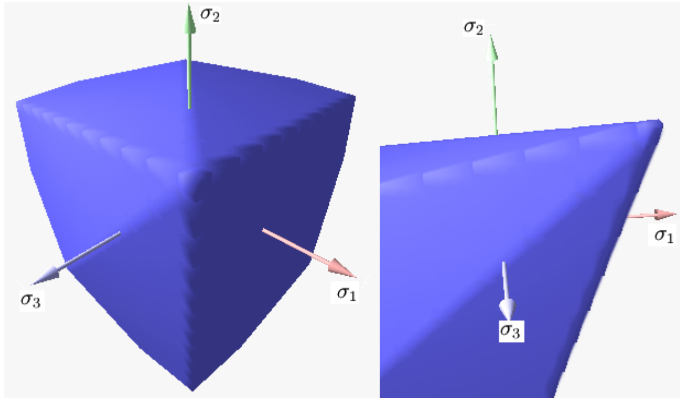


Figure 10: Mohr-Coulomb criterium

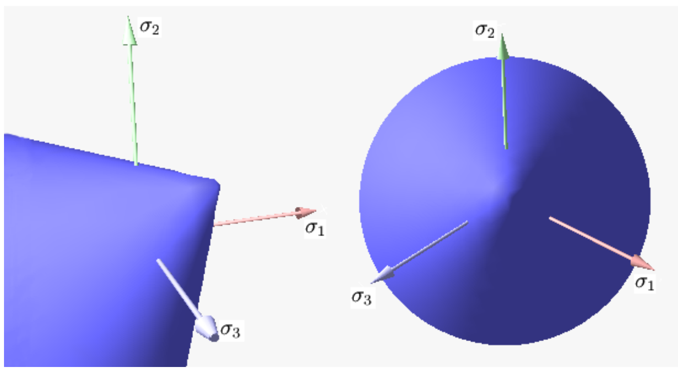


Figure 11: Drucker-Prager criterium

4.4.2 The Drucker Prager criterium [4] :

$$\sqrt{J_2} = A + BI_1$$

It can be expressed in term of the principal stresses :

Where A and B can be obtained from the limit of elasticity in tension $\sqrt{\frac{1}{6}\left((\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right)} = A + B\left(\sigma_1+\sigma_2+\sigma_3\right)$

In an uniaxial case we obtain $\sqrt{\frac{1}{3}\sigma_1^2} = A + B\sigma_1$

In tension, $\sigma_t > 0 \implies \sigma_t\left(\sqrt{\frac{1}{3}} - B\right) = A$

In compression, $\sigma_c < 0 \implies \sigma_c\left(-\sqrt{\frac{1}{3}} - B\right) = A$

So that :$A = \frac{2}{\sqrt{3}}\frac{\sigma_t\sigma_c}{\sigma_c-\sigma_t}$ and $B = \frac{1}{\sqrt{3}}\frac{\sigma_t+\sigma_c}{\sigma_t-\sigma_c}$

We can also express the A and B parameters in terms of the cohesion C and the friction angle Φ if we assume that the Drucker–Prager yield surface circumscribes the Mohr–Coulomb yield surface :

$A = \frac{6C\cos\Phi}{\sqrt{3}(3-\sin\Phi)}$ and $B = \frac{2\sin\Phi}{\sqrt{3}(3-\sin\Phi)}$

The drawing of Drucker Prager criterium in the principal stresses space is given in figure 11.

4.4.3 The Mazars criterium [8].

The Mazars criterium is often use for concrete like materials modelled with damage, like the previous criteria are used for plasticity models. The idea of mazars is that the non linearity of the material is generated by extension strains (i.e. postive strains).

$$\tilde{\varepsilon} = \sqrt{\langle\varepsilon_1\rangle^2 + \langle\varepsilon_2\rangle^2 + \langle\varepsilon_3\rangle^2}$$

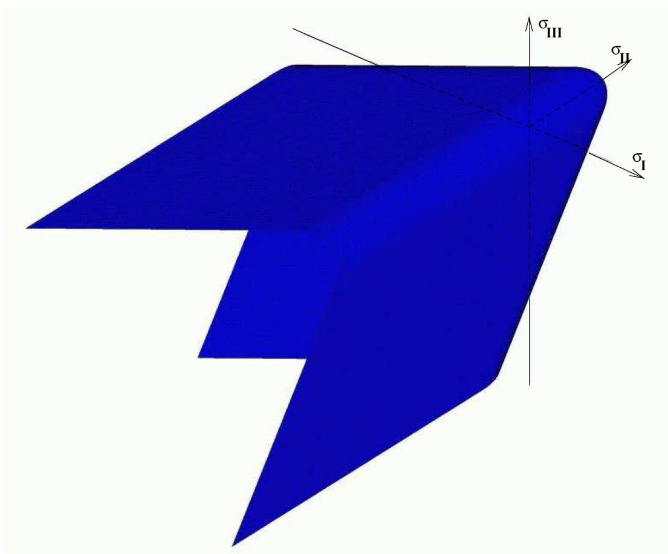


Figure 12: Mazars criterium

where ε_1, ε_2 and ε_3 are the principal strains and $\langle\rangle$ designs the Macauley brackets $\langle x \rangle = \begin{cases} x & if \quad x > 0 \\ 0 & if \quad x \leq 0 \end{cases}$. The threshold function is therefore writeen as following

$$\tilde{\varepsilon} - k(D) = 0$$

Where $k(D)$ is the yield strain depending of damage D. The inititial value is called ε_{d0} correspond to the limit of elasticity in positive strain.

The mazars criterium can be explicitely written in terms of strains and is widely used for damage models.

In the case of an uiaxial loading, $\overline{\overline{\sigma}} : \begin{bmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, then $\overline{\overline{\varepsilon}} : \begin{bmatrix} \frac{\sigma_{11}}{E} & 0 & 0 \\ 0 & \frac{-\nu\sigma_{11}}{E} & 0 \\ 0 & 0 & \frac{-\nu\sigma_{11}}{E} \end{bmatrix}$

The limit of elasticity in tension is given for a positive value of $\sigma_{11} = \sigma_t$, in that case $\varepsilon_{11} = \frac{\sigma_t}{E} > 0$ and $\varepsilon_{22} < 0$, $\varepsilon_{33} < 0$ this drives to $\tilde{\varepsilon} = \frac{\sigma_t}{E} = \varepsilon_{d0}$

$$\sigma_t = E\varepsilon_{d0}$$

In compression, the limit of elasticity is reached for $\sigma_{11} = \sigma_c < 0$, in that case $\varepsilon_{11} = \frac{\sigma_c}{E} < 0$ and $\varepsilon_{22} = \varepsilon_{33} = \frac{-\nu\sigma_c}{E} > 0$, this drives to $\tilde{\varepsilon} = -\sqrt{2}\nu\frac{\sigma_c}{E} = \varepsilon_{d0}$

$$\sigma_c = -\frac{E\varepsilon_{d0}}{\sqrt{2}\nu}$$

The drawing of the mazars criterium in the principal stresses space is given figure 12

5 Typical 1D non linear behaviour of materials

5.1 Plasticity

Plasticity is directly related to slips that can occur in the material by dislocation of grains, rearrangement of moleculs, slips along surfaces of decohesion [7]. The plasticity is characterized by a non linear behaviour linked to irreversible strains so called plastic strains. The total strain ε is splitted into elastic strains ε^e and plastic strains ε^p (Figure 13). The slope of the curve in the descending branch is identical to that of the ascending branch. The modulus of elasticity isn't affected by the "pure" plasticity. The stress intensity changing during the increasing of plastic strains is called strain hardening (softening in case of decreasing of stress).

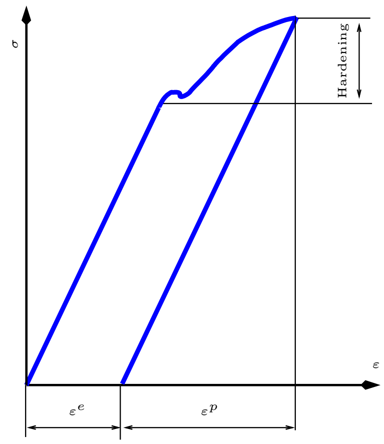
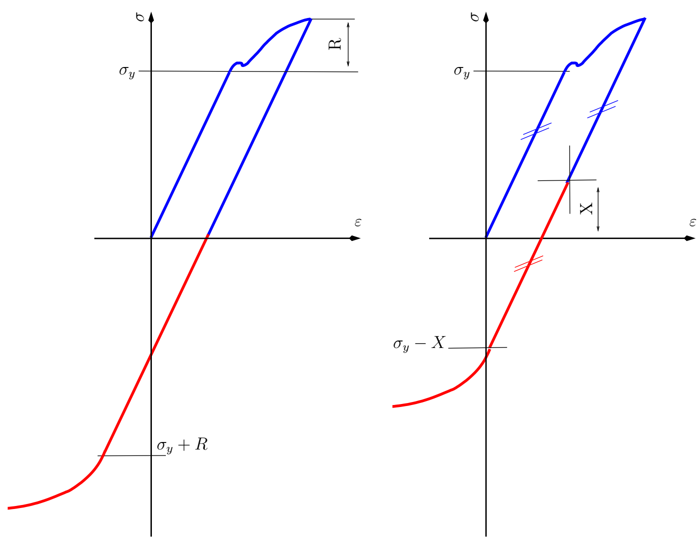


Figure 13: Plastic strains



(a) Isotropic hardening

(b) Kinematic hardening

Figure 14: Plastic hardening

$$\sigma = E\varepsilon^e = E\left(\varepsilon - \varepsilon^p\right)$$

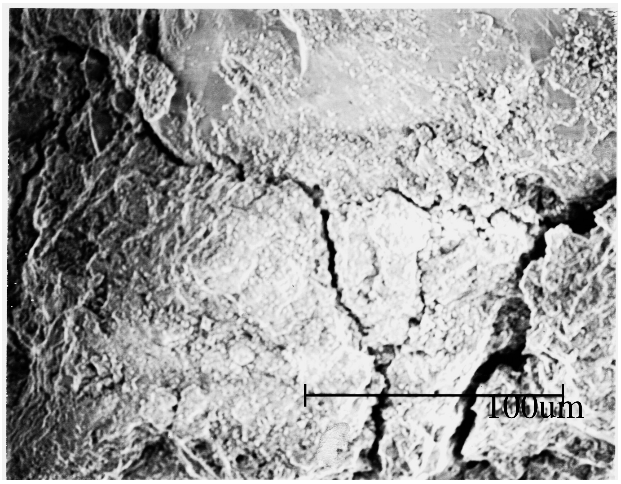
The hardening can be :

- Isotropic (Figure 14a). In this case the limit of elasticity is increased of the hardening value R and the elasticity criterium is $\sigma_{eq} - \sigma_y - R = 0$
- Kinematic (Figure 14b). The "center of elasticity" is moved to the hardening value X and the elasticity criterium is $(\sigma_{eq} - X) - \sigma_y = 0$
- A general case combines the two modes of hardening :

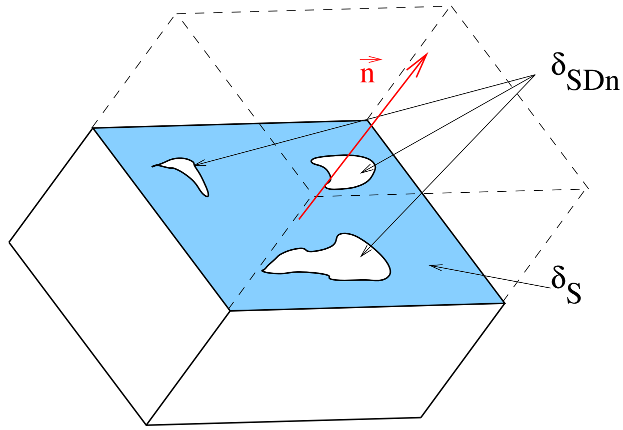
$$(\sigma_{eq} - X) - \sigma_y - R = 0$$

5.2 Damage :

The damage is linked to debonding of material and microcracking that accours at the mesoscopic level [7]. Lemaitre used an apple to represent the action of damage.



(a) Microcracking into concrete



(b) projection of defects on a plane

Figure 15: Meso-definition of damage after Lemaitre

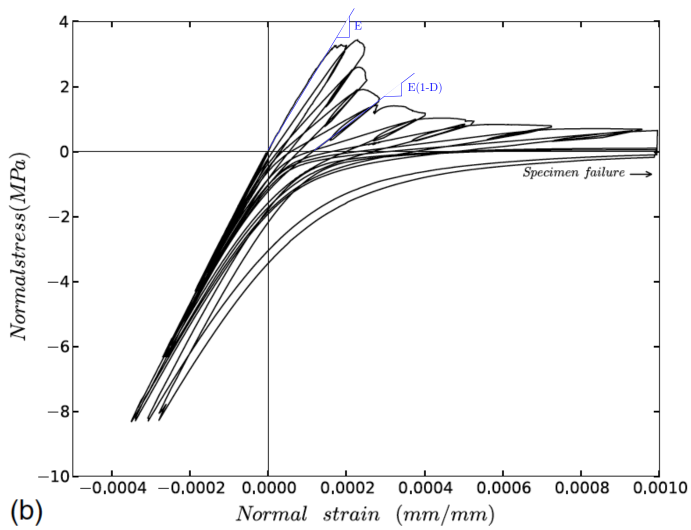


Figure 16: Stress-strain relation exhibiting damage after[9]

- Let δS be the intersection area of a given plane of normal $\vec{n}$ with a Representative Elementary Volume (RVE).
- Let δS_{Dn} be the effective area of micro-cracks and micro-cavities within the intersection plane at the point M
- The value of damage is then defined by

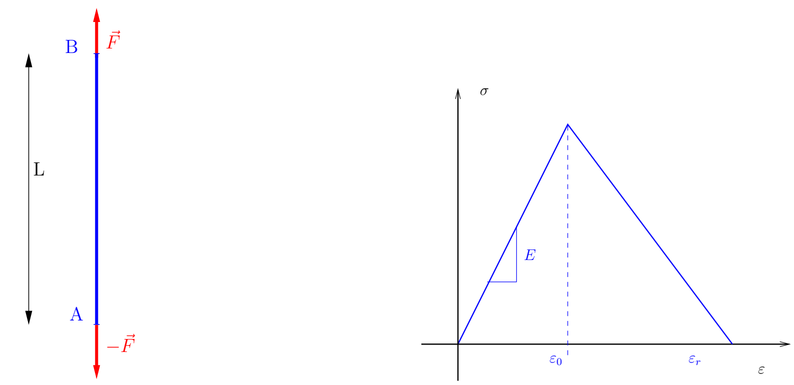
$$D(M, \vec{n}) = \frac{\delta S_{Dn}}{\delta S}$$

The damage D is by definition bounded between 0 and 1 : $0 < D < 1$

- $D = 0 \longrightarrow$ undamaged material
- $D = 1 \longrightarrow$ fully broken material

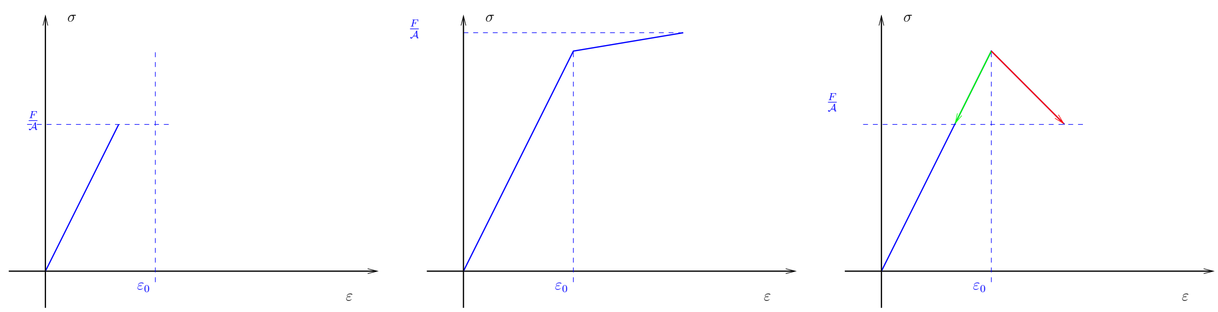
The damage affects the modulus of elasticity (Figure 16)

$$\sigma = E\,(1 - D)\,\varepsilon^e$$



(a) Geometry (b) Behavior of the material

Figure 17: Simple tension test



(a) Linear elastic (b) Strain hardening (c) Strain softening

Figure 18: Solutions depending on the nature of the material

6 Introduction to the strain localization problem:

6.1 1D tension test

When the material is exhibiting non linear behavior during a 1D tension test,

- if the stress is increasing with the strain, there is stress hardening
- if the stress is decreasing while the strain is increasing, there is tress softening.

Lets take the example of tension test of a simple 1D bar AB of section $\mathcal{A}$ illustrated at figure17a. The test is performed by increasing progressively the elongation δL of the specimen.

6.2 Solution of the problem

The balance equation drives to $\frac{\partial \sigma}{\partial x} = 0 \Rightarrow \sigma = \frac{F}{\mathcal{A}}$, that means that the stress is homogeneous over the bar.

The stress strain relationship (Figure 17b) can be written $\begin{cases} \varepsilon < \varepsilon_0 & \sigma = E\varepsilon \\ \varepsilon > \varepsilon_0 & \sigma = \frac{E\varepsilon_0}{\varepsilon_0 - \varepsilon_r}(\varepsilon - \varepsilon_r) \end{cases}$, the material can be either plastic or damageable.

In the case of linear elastic material, the solution for a given value of the load is unique (Figure 18a), as in the case of strain hardening (Figure 18b). For a softening material each material point has 2 solutions for a given stress (Figure 18c) the material can either stay elastic and folow the green path, or follow the non linear (red) path.

6.3 Homogeneous case

The simpler solution is to consider that the material stays homogeneous, i.e. the non linearities (plastic strains or damage) are the same a each point of the specimen. The load displacement behavior on the specimen is drawn in

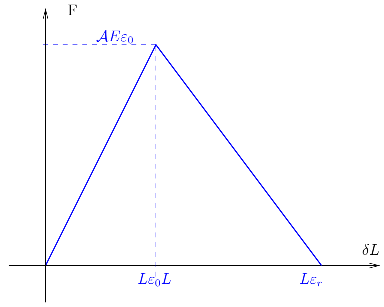


Figure 19: Load displacement solution, homogeneous case

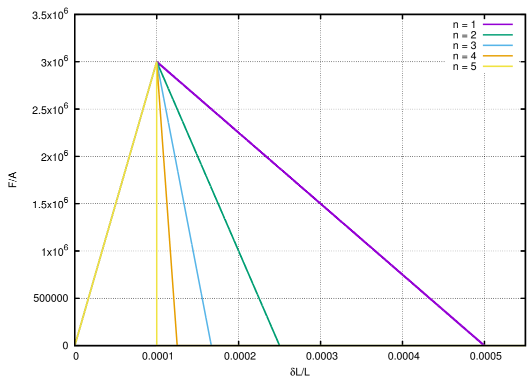


Figure 20: Load displacement solution, etherogeneaous cases

figure 19

6.4 Heterogeneous case

One can imagine that a part of the specimen is following the non linear (red) path while the other is following the linear (green) path. The balance equation is still verified as long as the stress in both parts are identical.

At a given stress $\sigma = \frac{F}{\mathcal{A}}$ the strains are :

- $\varepsilon = \frac{\sigma}{E}$ for the elastic part
- $\varepsilon = \varepsilon_r - \frac{\sigma}{E}\frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}$ for the non linear part

If the lengh of the non linear part is $\frac{L}{n}$, with $n \in \mathbb{N}, n > 1$, the lengh of the linear part is $L\frac{n-1}{n}$, the elongation is :

- $\delta L_e = \frac{F}{E\mathcal{A}} L \frac{n-1}{n}$ for the elastic part
- $\delta L_{nl} = \left(\varepsilon_r - \frac{F}{E\mathcal{A}}\frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}\right)\frac{L}{n}$ for the non linear part

The overall elongation becomes $\delta L = \frac{F}{E\mathcal{A}} L \frac{n-1}{n} + \left(\varepsilon_r - \frac{F}{E\mathcal{A}}\frac{\varepsilon_0}{\varepsilon_r - \varepsilon_0}\right)\frac{L}{n}$ and the the load displacement behavior is plotted figure 20for different values of n.

The result depends on the value of n, that means that the problem is hill posed. The energy disipated (integral of the load displacement relation) is decreasing and tends to zero with the size of the non linear element.

One solution consits of giving a characteristic length as the material paremeter which is the length of the non linear area. An other solution consits of using a softening parameter (here ε_r) which depends on the size of the element in order to disipate a constant value of energy.

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