On the valuation of constant barrier options under spectrally one sided exponential Lévy models and Carr's approximation for American Puts

by

Florin Avram [1], Terence Chan [2] and Miguel Usabel [3]

Abstract: This paper provides a general framework for pricing options with a constant barrier under spectrally one-sided exponential Lévy model, and uses it to implement of Carr's approximation for the value of the American put under this model. Simple analytic approximations for the exercise boundary and option value are obtained.

Keywords: American options; perpetual approximation; spectrally negative exponential Lévy process

1 Introduction

This paper develops Carr's "Erlang approximation" for the price of an American put option under the spectrally negative exponential Lévy model.

There has been lots of recent interest in mathematical finance (see for example [10]) towards extending results based on the exponential Brownian motion model to results based on exponential Lévy models. This is motivated by the superior fits to the data and hence improved pricing formulas and hedging strategies provided by several special classes of Lévy models – see for example the hyperbolic model of Eberlein and Keller [9], the variance-gamma model of Madan [19] and the inverse Gaussian model of Barndorf-Nielsen [2]. Following previous work of Gerber and Shiu [13] and Gerber and Landry [12], we investigated the Lévy pricing of American puts, starting with the considerably easier case of spectrally one sided Lévy processes. The results for these processes are different depending on whether the barrier can only be crossed "continuously" or also by jumps, the first case being very

[1]Dept. of Actuarial Maths. & Statistics, Heriot-Watt University, Edinburgh EH14 4AS U.K.,E-mail:florin@ma.hw.ac.uk

[2]Dept. of Actuarial Maths. & Statistics, Heriot-Watt University, Edinburgh EH14 4AS U.K., E-mail:terence@ma.hw.ac.uk

[3]Edif. Miguel de Unamuno, Universidad Carlos III, Avda. Universidad Carlos III 22, 28270 Colmenarejo (Madrid), Spain, E-mail: usabel@emp.uc3m.es

similar to the classical Brownian case. We have chosen therefore to work on the latter more complicated case of pricing American puts under spectrally negative exponential Lévy models. Note that this clearly requires knowing the distribution of the "overshoot" below the barrier, and convenient formulas for this are available in the spectrally negative case which relieving one of the need to perform a Wiener Hopf factorization. The general case of Lévy processes which may jump in either direction will require further:

1) Either obtaining Wiener-Hopf factorizations in the special cases mentioned above, or developing numerical approximations for these (as employed for example in analogue problems in queueing theory).

2) Modifying the "early exercise decomposition" as accumulated interest on the strike price while below the barrier due to Kim [14], Jacka [15] and Carr, Jarrow and Myneni [6], so as to include the effect of jumps back over the barrier identified by Pham [23].

The pricing of American put options, with payoff

$$e^{-r\tau}\,(K - S_{\tau})_{+}$$

where $\tau = \min(\tau_*, T)$, T is the expiration time and τ_* denotes the hitting time of an optimal exercise boundary B_s, $0 \le s \le T$, is quite challenging due the fact that the optimal exercise boundary depends on time. More precisely, the optimal exercise point at time s is some function of the remaining time $T - s$ until expiration. An excellent overview of the problem may be found for example in the book of Kwok [18].

To remove the dependence of the barrier on time, Samuelson and H. McKean [20] proposed already in 1965 in what was maybe the first paper in mathematical finance to approximate the problem by that of pricing a "perpetual" option (with infinite expiration time $T = \infty$). For the perpetual, the dependence on the remaining time disappears, leaving us with the problem of valuation of a barrier put with fixed constant barrier B, followed by the optimization of B. By this approach, perpetual American options were recently priced under the spectrally positive/negative exponential Lévy models by Gerber and Shiu [13] and Gerber and Landry [12], and under the general exponential Lévy model by Boyarchenko and Levendorskii [4]; the price is simply the average of the discounted put payoff with respect to the distribution of the price after the barrier crossing. Note however that perpetual approximations are generally too crude. More refined analytic approximations have been proposed by McMillan [21] (who also removes the boundary's dependence of time by a transformation) and Omberg [22] (who looked for the best exponential exercise boundary).

Recently, a new method for removing the dependence on time has been proposed by P. Carr [7]. The first step consists in replacing the perpetual by an option with random exponential expiration time $\tilde{T}$, with $\mathbb{E}\tilde{T} = T$, which was called a "Canadian option". The exponential assumption has the virtue of making the optimal exercise boundary independent of time, just like in the classical perpetual case and so this step is as easy to implement as McKean's perpetual approximation. The Canadian approximation was then further refined by considering options whose expiration time $\tilde{T}$ is a sum of n exponentials and thus $\Gamma_{n,n/T}$ distributed (with $\mathbb{E}\tilde{T} = T$); this will be referred to as "n-Erlangian" options. The value of an n-Erlangian option may be computed via an iterative method with n stages. When $n \to \infty$, this approximation converges to the true value.

Carr's method retains some of the simplicity of the perpetual approximation and at the same time has the virtue of converging to the true value. Already the first stage "Canadian" approximation leads to a simple **time dependent** analytic answer for the put exercise barrier (see (16) below) which is exact for both very short and very long expiration times (in the limit).

Our paper extends Carr's results for American put options to the spectrally negative exponential Lévy model. To achieve that, we needed to develop a general approach for pricing "Canadian" barrier options under this model.

1.1 Literature survey

Besides McMillan's, Omberg's and Carr's approaches, some other very successful recent approximation methods under the exponential Brownian motion model are due to Broadie and Detemple [5], Ju [16] (who extended Omberg's approach to a multistage procedure) and Ju and Zhong [17] (who provided a second order perturbation approximation).

In the context of American options under the exponential Lévy model, X.L. Zhang [26], [28] extended the approach of McMillan [21] and H. Pham [23] extended the representation of the American early exercise premium due to Kim [14], Jacka [15] and Carr, Jarrow and Myneni [6].

Gerber and Shiu [13] showed that analytical formulas for the exercise barrier of perpetual options may be obtained under certain exponential Lévy models, since they depend on knowing the Laplace transform of the value function rather than on the value function itself. Their first results were obtained under the assumption of a pure jump spectrally one sided Lévy process (the classical risk process), which allows one to use the machinery of renewal equations. Later, Gerber and Landry [12] incorporated a Brownian component and Chan [8] extended the results to the case of general spectrally negative Lévy processes.

Note: Extending these results to Lévy processes with jumps of both signs is a problem of great practical interest [19], but considerably more complicated. Several results, like Proposition 2.1, Bingham's theorem 6b) (which is based on the Wiener Hopf factorization) used in our Section 3 and Pham's representation of the American early exercise premium change substantially in the general case. Previous work in the general Lévy case was based on two special representations of Lévy processes: one as time changed Brownian motions and the other as additive functionals of Brownian motion evaluated at the inverse of the local time at 0–see [10], [24].

1.2 The exponential Lévy model

We consider below assets modelled by an exponential Lévy process of the form

$$\begin{aligned} S_t &= e^{Y_t} && (1) \\ Y_t &= y + \mu\, t + \sigma W_t - J_t. && (2) \end{aligned}$$

where W_t is a standard Brownian motion and J_t is an independent Lévy jump process with positive jumps; thus, Y_t is a spectrally negative Lévy process. We will assume that the Lévy measure of J_t ν satisfies the simplifying assumption

$$\int_0^1 x\, \nu(dx) < \infty. \tag{3}$$

which ensures that J has finite variation. [4]

[4]The condition (3) is assumed purely for convenience, to enable us to avoid excessively messy expressions. The extension of the results described in the sequel to Lévy processes with infinite

If J is a compound Poisson process $\sum_{i=1}^{N_t} Z_i$, where N_t is a Poisson process with intensity λ and the jumps Z_i are i.i.d random variables independent of N_t with a distribution $P(dx)$ concentrated on the positive axis, then $\nu(dx) = \lambda P(dx)$ and (3) is automatically satisfied. [5]

Because the jumps of J are all positive, the moment generating function $\mathbb{E}[e^{\theta(Y_t-y)}]$ exists for all $\theta \geq 0$ and

$$\mathbb{E}[e^{\theta(Y_t-y)}] = e^{t\,c(\theta)} \tag{4}$$

for some function c, which is referred to as the cumulant generating function. Under the simplifying assumption (3), the cumulant generating function is given by:

$$c(\theta) = \frac{\sigma^2\theta^2}{2} + \mu\theta + \int_0^\infty (e^{-\theta x} - 1)\,\nu(dx) = \frac{\sigma^2\theta^2}{2} + \mu\theta + \nu^*(\theta) \tag{5}$$

where we denoted by $\nu^*(\theta) = \int_0^\infty (e^{-\theta x} - 1)\,\nu(dx)$ a "Laplace transform" of the jump measure $\nu(dx)$ appropriately modified to allow for the case when it has infinite mass around the origin (in which case $\nu^*(\theta)$ is still well defined by (3)).

By the fundamental theorem of asset pricing, to avoid arbitrage it is necessary to use a risk neutral pricing measure for options. If the observed stock prices follow an exponential Lévy model, the risk neutral pricing measure is not unique and the issue of choosing between the possible risk neutral measures is quite delicate. A choice recommended for its simplicity is to use the "Esscher transformed measure", i.e. an exponential Lévy model which is risk neutral and differs from the observed measure only by its drift. Assuming that the asset continuously pays dividends at rate q, this is tantamount to assuming the condition (see [11]):

$$c(1) = r - q \tag{6}$$

where r is the discount rate and q is the dividend rate. (In the presence of dividends, the process $e^{-(r-q)t} S_t$ has to be a martingale under the risk neutral measure). The condition (6)

variation is essentially trivial and is left to the interested reader – one must compensate for the small jumps in integrals with respect to ν, resulting in rather messier algebra.

[5] The extension from compound Poisson to general Lévy processes is quite important in view of the fact that the current "favourites" for asset prices modelling, the hyperbolic and the variance gamma process have unbounded Lévy measure at 0.

means that the drift must satisfy

$$\mu = r - q - \sigma^2/2 - \nu^*(1), \tag{7}$$

so we are working only with cumulant generating functions of the form:

$$c(\theta) = \frac{\sigma^2}{2}(\theta^2 - \theta) + \int_0^\infty (e^{-\theta x} - e^{-x})\,\nu(dx) + (r-q)\theta.$$

1.3 The "Black Scholes" equation for barrier options

We consider below "down" barrier options, which may earn either a **final payoff** $f(Y_T)$ at the expiration time T or a rebate or **boundary payoff** $\pi(Y_{\tau_B})$ at the first time τ_B when the asset process Y_t drops below an exercise boundary B, which may depend on time. The example we are interested is that of the American put option, in which case the payoff and rebate have the same functional dependence given by $f = \pi = (K - S_t)_+ = (K - e^{Y_t})_+$. However, the Erlang stages approximation method we use forces us eventually to consider options with arbitrary final payoffs $f(y)$ (the boundary "put" payoff $\pi = (K - S_t)_+$ will stay fixed).

Let $v_{B,\pi,f,r}(t, y) = v(t, y)$ denote the arbitrage free value of this option under the complete exponential Brownian motion model, expressed in terms of the value y at time t of the logarithm of the price Y_t. Then

$$v_B(t, y) = \mathbb{E}^*_{\{Y_t = y\}}\left[e^{-r(T-t)} f(Y_T)\, 1_{\{\tau_B \geq T\}} + e^{-r(\tau_B - t)}\, \pi(Y_{\tau_B})\, 1_{\{\tau_B < T\}}\right] \tag{8}$$

where $\mathbb{E}^*$ denotes risk neutral expectation.[6]

Under the unique dividends adjusted risk neutral exponential Brownian motion model, $v(t, y)$ satisfies the backward Kolmogorov equations:

$$\begin{aligned} &\Gamma v - rv + \frac{\partial v}{\partial t} = 0 &&\text{for } y \geq B_t \\ &v(t, y) = \pi(y) &&\text{for } y \leq B_t \\ &v(T, y) = f(y) \end{aligned} \tag{9}$$

where Γ, the infinitesimal generator of Y is $(\Gamma f)(u) = \frac{\sigma^2}{2} f''(u) + \mu f'(u)$ and $\mu = r - q - \sigma^2/2$.

[6] European options are the particular case in which there exists only a final payoff f and $\pi = 0$.

We will actually work with the forward Kolmogorov equations in which the current time variable t is replaced by the remaining time until expiration $s = T - t$, yielding:

$$
\begin{aligned}
&\Gamma v - rv - \frac{\partial v}{\partial s} = 0 && \text{for } \ y \geq B_s \\
&v(s,y) = \pi(y) && \text{for } \ y \leq B_s \\
&v(0,y) = f(y)
\end{aligned}
\tag{10}
$$

If the stock price is modelled as an exponential Lévy model, $v(s,y)$ satisfies the same equation (10), but with the generator modified to:

$$
(\Gamma f)(u) = \frac{\sigma^2}{2} f''(u) + \mu f'(u) + \int_0^\infty [f(u-z) - f(u)]\, \nu(dz). \tag{11}
$$

(where μ satisfies the risk neutrality assumption (7)).

1.4 Canadian options

The idea of Canadian options is to consider options with an exponentially randomized expiration time $T^{(\alpha)}$ of some arbitrary but fixed rate α. Due to the memoryless property of the exponential, this remove the dependence on the starting time in the value of Canadian options.

Let $V_{B,\pi,f,r,\alpha}(y) = V(y)$ denote the value of the "Canadian barrier option" with payoffs π, f, defined as:

$$
V(y) = \mathbb{E}^*_{\{Y_0=y\}} \left[e^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})\, 1_{\{\tau_B \geq T^{(\alpha)}\}} + e^{-r(\tau_B)}\, \pi(Y_{\tau_B})\, 1_{\{\tau_B < T^{(\alpha)}\}} \right] \tag{12}
$$

The dependence on B, π, f, r, α will be usually suppressed. If $v(s,y)$ denotes the value of the corresponding barrier option with the same payoffs, then the Canadian value $V(y)$ is obtained simply by averaging $v(s,y)$ with an exponential distribution for the remaining time s.

$$
V(y) = \int_0^\infty \alpha e^{-\alpha s} v(s,y) ds
$$

This only differs from the Laplace transform in s by the extra factor α and is also known as a Laplace-Carson transform.

Another essential simplification when valuing Canadian American options is provided by the memoryless property of the exponential distribution, which entails that their **optimal exercise boundaries have to be constant. As such, valuation reduces to a one dimensional optimization over the possible barriers** a.

Consider therefore the value $V_{a,\pi,f,r,\alpha}(y) = V(y)$ of a Canadian barrier option with fixed barrier a. Taking the Laplace-Carson transform in (10) changes the differentiation operator in t into multiplication, yielding the integro-differential equation:

$$\begin{aligned} (\Gamma V)(y) - \delta V(y) + \alpha f(y) = 0 &\text{ for } y \geq a \\ V(y) = \pi(y) \qquad\qquad\qquad &\text{ for } y \leq a \end{aligned} \tag{13}$$

where we put $\delta = r + \alpha$ (which may be interpreted as a combined "killing" rate). This equation is of course considerably simpler than its time dependent counterpart.

Note: The payoff of a standard American put is given by $\pi(y) = (K - e^y)_+$. Since we will obviously only choose constant exercise levels $a < \log K$, the boundary condition in (13) may be simplified to $V(y) = \pi(y) = K - e^y$ for $y \leq a$. This simplifies many of the subsequent calculations and will be repeatedly used without further explicit mention.

Example: The Canadian value of a digital option $I_{\{Y_{T(\alpha} \in (y,y+dy)\}}$ which earns a discounted currency unit if at the exponentially distributed expiration Y_T is near y is:

$$h(y) = \int_0^\infty \alpha e^{-(\alpha+r)s} \mathbb{P}(Y_s \in dy)\, ds$$

Formally, this solves the previous equation (13) with $f(x) = \delta_y(x), a = -\infty$. Up to the constant α, $h(y)$ is also called the resolvent measure of the Lévy process Y or Green function; its importance stems from the fact that the value of any other Canadian "European" option (i.e. an option which has only final payoff and no barrier) may be obtained simply by integrating the final payoff $f(y)$ multiplied by the kernel $h(y - x)$. In fact, the value of Canadian barrier options may also be represented in terms of the resolvent – see Proposition 4.3.

1.5 The early exercise decomposition

It has been often found convenient to replace the "nonhomogeneous" equation (13) by an "homogeneous" equation with no final payoff f, by subtracting out of V the "Canadian European" value of the final payoff

$$\tilde{f}(y) = \mathbb{E}_y[e^{-rT^{(\alpha)}} f(Y_{T^{(\alpha)}})] \tag{14}$$

which satisfies the same nonhomogeneous equation as V, and thus the difference $\tilde{V} = V - \tilde{f}$ satisfies the homogeneous equation. We find:

$$\begin{aligned} &(\Gamma\tilde{V})(y) - \delta\tilde{V}(y) = 0 \\ &\tilde{V}(y) = \tilde{\pi}(y) \quad \text{for } y \leq a \\ &\tilde{V}(\infty) = 0 \end{aligned} \tag{15}$$

where $\tilde{\pi}(y) = \pi(y) - \tilde{f}(y)$.

The financial interpretation is that we split the value of the option in two parts: the "Canadian" value of the final payoff and the **early exercise premium** $\tilde{V}$. Moreover, $\tilde{V}$ can be interpreted as the value of a perpetual American option with payoff $\tilde{\pi}$. This idea has appeared in several places in the literature (see [8], [21] and [28]).

1.6 Other stationary approximations

Most of the other approximations for the time dependent problem (9) (usually for "early exercise" part) seem to result in the same type of equation as (15).

Zhang(95) proposed the "one step discretization" approximation for the "early exercise" premium. For an interval of size T, this leads to equation (15) with $\delta = r + \frac{1}{T}$ and $\tilde{f}$ being the European value given by $\tilde{f}(T, y) = \mathbb{E}_y[e^{-rT} f(Y_T)]$.

McMillan(87) proposed to approximate the "early exercise" premium as $\tilde{v}(t, y) \approx h(t)\tilde{v}(y)$, where $h(t) = 1 - e^{-r(T-t)}$ was judiciously chosen to combine the terms $-rh + h'$ in the resulting equation and to equal 0 at expiration. This leads again to equation (15) where

$\delta = \frac{r}{1-e^{-rT}}$ and $\tilde{f}$ is again the initial European value given by $\tilde{f}(T,y) = \mathbb{E}_y[e^{-rT} f(Y_T)]$. We note that in the classical Brownian case studied by McMillan, the ODE (15) has for fixed a the explicit solution $\tilde{V} = \tilde{\pi}(a)e^{\theta(y-a)}$, where θ is the negative root of the quadratic equation $\sigma^2/2\theta^2 + \mu\theta - \delta = 0$. Hence, the optimal exercise barrier is found quite easily by maximization with respect to a, leading to the nonlinear equation $\tilde{\pi}'(a) = \theta\tilde{\pi}(a)$.[7]

1.7 Outline of main results

In section 2 below we provide in Proposition 2.1 an analytic expression for the Laplace transform (with respect to the initial price) of the value of barrier options with exponential expiration. This result generalizes classical risk theory results on computing "multivariate ruin probabilities" –see Corollary 2.5 and for comparison theorem 6 a) of Bingham [3], Gerber and Shiu [13], Gerber and Landry [12] and Usabel [25]. Essentially, it reduces the risk neutral valuation of any barrier option under spectrally one sided Lévy models to carrying out some integrations and inverting Laplace transforms.

In Sections 3 we determine the optimal exercise barrier for "Canadian American" options (directly from the Laplace transform, avoiding the inversion). As an illustration of the elegance of the results obtained under the spectrally one sided model, we highlight now the time dependent analytic approximation of the exercise boundary (30) obtained already by the Canadian approximation. This reduces to solving two algebraic equations in the presence of dividends and in the absence of dividends to only one explicit formula for the boundary L^*:

$$\boxed{L^* = K\big((\delta_+ - 1)rT\big)^{\frac{1}{\delta_+}}} \tag{16}$$

where δ_+ denotes the largest solution of the equation: $c(\theta) = \delta = r + T^{-1}$, $c(\theta)$ is the cumulant generating function of the process and T denotes the remaining time until expiration (this

[7] While we focused in this paper on the "Canadian" approximation for removing time dependence, we note that since the discretization and McMillan approximations lead to the same type of time independent equation (15) (which is a particular case of (13)), similar results for these approximations can be obtained. For McMillan's approximation, we need only to replace δ with $\frac{r}{1-e^{-rT}}$ and to replace the Canadian value of the final payoff by the European value of the final payoff. In principle, these methods which use the exact European value and only approximate the early exercise part are expected to work better, but further work to confirm this is necessary.

generalizes the formula obtained by Carr for the diffusion model).

While the barrier (16) is only optimal for Canadian American options with exponential expiration of expected duration T, we propose to use it also as an approximate formula for the optimal exercise boundary of usual American options with remaining lifetime T (with T decreasing to 0); this provides a time dependent approximation for the exercise boundary which is shown to be asymptotically exact when $T \to \infty$ and $T \to 0$ (see Corollary 3.1d,e and the Remark 3 following it) [8]

In Section 4 we identify in Theorem 4.1 and Corollary 4.2 some classical fluctuation theory identities which allow us to invert the Laplace transform in terms of the resolvent density of the Lévy process. The latter is available explicitly for compound Poisson processes with a phase-type distribution of jumps and numerically in general. The end result, Proposition 4.3 is an explicit formula for Canadian put options in terms of either the resolvent (more convenient numerically) or the Wiener Hopf factors of the process. Since the resolvent may be computed numerically via Fast Fourier transform–see [8], this approach may be preferable numerically to the inversion of the Laplace transform.

In Section 5 we consider n-Erlangian options. We discuss two possible recursive implementations: one based on using Laplace transforms only, which reduces to solving algebraic equations, and the second based on computing the resolvent. We also establish in Corollary 5.2 the convergence of the N-Erlangian approximation to the finite expiration price.

Finally, in the Appendix we compute the value of Canadian puts in the exponential jumps case.

We illustrate now the importance of using statistically appropriate models by displaying the approximate exercise barrier as a function of the remaining time $0 \leq t \leq 10$ when the log price process is compound Poisson with exponential jumps of distribution $\beta e^{-\beta x}$; the dotted lines display the exercise barrier for a Gaussian approximation of this process of equal given variance $\sigma^2 t$ (this determines the intensity of the jumps via $\lambda\beta^{-2} = \sigma^2$). We

[8]The performance of this first approximation is comparable with that of the generalization of McMillan's approximation proposed by Zhang [28]. We expect that even these simple approximations might have a quite significant impact on pricing American options in practice, since we believe that taking advantage of the superior numerical fit of the exponential Lévy model to real data may be more important than improving the accuracy of the exercise boundary.

Figure 1: **Exercise Barrier approximations, Lévy and Gaussian approx.,** $\beta = 20$

Figure 2: **Exercise Barrier approximations, Lévy and Gaussian approx.,** $\beta = 1$

took $K = 10$ and $r = .1 = \sigma^2/2$ (this fixes the long time limit of the Gaussian barrier to $L = K\frac{1}{1+\frac{\sigma^2}{2r}}$ to $\frac{K}{2=5}$). We note that while for the first value $\beta = 20$ (Fig. 1) the barrier is very close to its Gaussian approximation, for $\beta = 1$ (Fig. 2) the difference is considerable, as both barriers approach quickly their different respective long time limits of $K\frac{1}{1+\frac{\sigma^2}{2r}\frac{1}{(1+\beta^{-1})^2}} = 8$ and $K\frac{1}{1+\frac{\sigma^2}{2r}} = 5$.

This example emphasizes the rather obvious point that underparametrized models like either Brownian motion or spectrally one sided compound Poisson models may be quite inadequate for approximating more general processes and hence the desirability of developing models which are rich enough to capture market behaviour (and hopefully still admit convenient pricing and hedging formulas).

2 The Laplace transform with respect to the initial price of the value of Canadian barrier options

We will consider from now Canadian barrier options with a **fixed** barrier $B_t = a$ and value $V(y)$ (note that we are suppressing in the notation the dependence on a, π, f, r, α). It will be convenient here to work with the "surplus over the barrier" variable $u = y - a$ which moves the barrier to $u = 0$. We denote by $V_a(u)$ the function defined by the "shift" $V_a(u) = V(a+u) = V(y)$ (similarly, the "shifted" payoffs f, π are $f_a(u) = f(u+a), \pi_a(u) = \pi(u+a)$).

We obtain now the Laplace transform of the value function above the barrier, by which we mean the usual Laplace transform with respect to the surplus variable

$$V_a^*(\theta) = \int_a^\infty e^{-\theta(y-a)} V(y)\, dy = \int_0^\infty e^{-\theta u} V_a(u)\, du, \tag{17}$$

for functions satisfying the equation (13) (or (15), which is a particular case of 13).

Let $f_a^*(\theta) = \int_0^\infty e^{-\theta u} f(u+a))\, du = e^{\theta a} \int_a^\infty e^{-\theta y} f(y) dy$ and $\nu_x^*(\theta) = e^{\theta x} \int_x^\infty e^{-\theta y} \nu(dy)$ denote the "over the barrier Laplace transforms" of $f_a(y)$ and $\nu_x(dy)$.

Taking Laplace transforms in (13), we obtain the following key result, similar to a result of Usabel [25].

Proposition 2.1. *a) The Laplace transform $V_a^*(\theta)$ is given by*

$$V_a^*(\theta) = \frac{\alpha\big(f_a^*(\theta) - f_a^*(\delta_+)\big) + \int_0^\infty \pi(a-x)\big(\nu_x^*(\theta) - \nu_x^*(\delta_+)\big)\, dx - \frac{\sigma^2}{2}\pi(a)(\theta - \delta_+)}{\delta - c(\theta)} \tag{18}$$

where δ_+ denotes the largest (necessarily non-negative) root of $c(\theta) - \delta = 0$.

b) In the pure jump case, the value of a Canadian barrier option with 0 initial surplus is:

$$\mu V_a(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x)\nu_x^*(\delta_+)\, dx \tag{19}$$

c) In the jump-diffusion case $V_a(0+) = \pi(a)$ and

$$\frac{\sigma^2}{2} V_a'(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x)\nu_x^*(\delta_+)\, dx - (\frac{\sigma^2}{2}\delta_+ + \mu)\pi(a) \tag{20}$$

Note: The equation $c(\theta) - \delta = 0$ (which generalizes McMillan's quadratic equation obtained in the pure diffusion case) has always at most 2 solutions, since the Laplace exponent c is convex. Note that when $\delta = 0, \theta = 0$ is always a solution; let θ_0 denote the largest solution in this case. Note that $\theta_0 > 0$ if and only if $c'(0) = \mathbb{E}(Y_1) < 0$. Also, $c(\theta) \leq 0$ for $\theta \in [0, \theta_0]$ while c is increasing over the interval $[\theta_0, \infty)$; therefore, $c(\theta)$ has a unique continuous inverse $\gamma(\delta) \geq \theta_0$ which is defined for $\delta \geq 0$ and satisfies $c(\gamma(\delta)) = \delta$ for all $\delta > 0$ and $\gamma(c(\theta)) = \theta$ for $\theta \geq \theta_0$. Hence, $c(\theta) - \delta = 0$ has always a unique positive solution $\delta_+ = \gamma(\delta)$.

Proof. Noting that $V_a(u - x) = \pi(a + u - x)$ for $x \geq u$, we take the Laplace transform with respect to the surplus variable u in (13) to obtain

$$
\begin{aligned}
0 &= \frac{\sigma^2}{2}\big(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))\big) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta) \\
&\quad + \alpha f_a^*(\theta) + \int_{x=0}^{\infty}\int_{u=0}^{\infty} e^{-\theta u}[V_a(u-x) - V_a(u)]\, du\, \nu(dx) \\
&= \frac{\sigma^2}{2}\big(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))\big) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta) \\
&\quad + \alpha f_a^*(\theta) + \int_0^{\infty}\left\{\int_0^x e^{-\theta u}\pi(a+u-x)\, du + \int_x^{\infty} e^{-\theta u} V_a(u-x)\, du - V_a^*(\theta)\right\}\nu(dx) \\
&= \frac{\sigma^2}{2}\big(\theta^2 V_a^*(\theta) - (V_a'(0+) + \theta V_a(0+))\big) + \mu(\theta V_a^*(\theta) - V_a(0+)) - \delta V_a^*(\theta) \\
&\quad + \alpha f_a^*(\theta) + \int_0^{\infty}\left\{e^{-\theta x}\int_{-x}^0 e^{-\theta z}\pi(a+z)\, dz + (e^{-\theta x} - 1)V_a^*(\theta)\right\}\nu(dx) \\
&= c(\theta)V_a^*(\theta) - \delta V_a^*(\theta) + \alpha f_a^*(\theta) - \frac{\sigma^2}{2}\big(V_a'(0+) + \theta V_a(0+)\big) - \mu V_a(0+) \\
&\quad + \int_0^{\infty} e^{-\theta x}\int_{-x}^0 e^{-\theta z}\pi(a+z)\, dz\, \nu(dx).
\end{aligned}
$$

Rearranging the above gives

$$
\begin{aligned}
V_a^*(\theta) = \big(\delta - c(\theta)\big)^{-1}\Big\{ &- \mu V_a(0+) - \frac{\sigma^2}{2}\big(V_a'(0+) + \theta V_a(0+)\big) \\
&+ \alpha f_a^*(\theta) + \int_0^{\infty} e^{-\theta x}\int_{-x}^0 e^{-\theta z}\pi(z+a)\, dz\, \nu(dx)\Big\}.
\end{aligned}
\tag{21}
$$

Changing the order of integration and the integral variable to $x = z - u, z$ with $0 \leq x \leq \infty, z \geq x$ we find:

$$
\int_0^{\infty} e^{-\theta x}\int_{-x}^0 e^{-\theta z}\pi(z+a)\, dz\, \nu(dx) = \int_0^{\infty} \pi(a-x)\nu_x^*(\theta)\, dx. \tag{22}
$$

The right-hand side of (21) still involves certain unknown quantities, namely $V_a'(0+)$ and $V_a(0+)$. At this point, we need to consider two different cases separately.

Pure jump model:

Suppose the logarithmic stock price is pure jump and has no Brownian component, so that $\sigma = 0$. In this case, we need only find $V_a(0+)$. Consider the value of the right-hand side of (21) at the largest (necessarily non-negative) root δ_+ of $c(\theta) - \delta = 0$. Since the Laplace transform is well-defined for all $\theta \geq 0$, it must in particular be finite at δ_+, implying thus the equation (19): $\mu V_a(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x)\nu_x^*(\delta_+)\,dx$.

Substituting the above and (22) into (21) establishes (18).

Diffusion-jump model [9]

If a Brownian component is present, so that $\sigma \neq 0$, $u \mapsto V_a(u)$ is always continuous at 0 (see [8]), so we know that $V_a(0+) = \pi(a)$. To find an expression for $V_a'(0+)$, consider again the value of the right-hand side of (21) at δ_+; we obtain now (20):

$\frac{\sigma^2}{2}V_a'(0+) = \alpha f_a^*(\delta_+) + \int_0^\infty \pi(a-x)\nu_x^*(\delta_+))\,dx - (\frac{\sigma^2}{2}\delta_+ + \mu)\pi(a)$. Substituting this and (22) into (21) establishes (18). $\square$

We consider now the put boundary payoff $\pi(y) = (K - e^y)_+ = K - e^y$, starting with the preliminary case $\pi(x) = e^{\eta x}$ (to be applied with $\eta = 0$ and $\eta = 1$).

Lemma 2.2. *If $\pi(x) = e^{\eta x}$, then:*

$$\int_0^\infty \pi(-x)\nu_x^*(\theta)\,dx = \int_0^\infty e^{-\eta x}\nu_x^*(\theta)\,dx = \frac{\nu^*(\eta) - \nu^*(\theta)}{\theta - \eta} \tag{23}$$

where $\nu^*(\theta) = \int_0^\infty (e^{-\theta x} - 1)\,\nu(dx)$.

Proof.

$$\int_0^\infty \pi(-x)\nu_x^*(\theta)\,dx = \int_0^\infty e^{-\eta x}\nu_x^*(\theta)\,dx = \int_0^\infty e^{-\eta x}\int_x^\infty e^{\theta(x-y)}\nu(dy)$$
$$= \int_0^\infty e^{-\theta y}\frac{e^{(\theta-\eta)y-1}}{\theta-\eta}\nu(dy) = \int_0^\infty \frac{e^{-\eta y} - e^{-\theta y}}{\theta - \eta}\nu(dy) = \frac{\nu^*(\eta) - \nu^*(\theta)}{\theta - \eta}$$

Applying the formula (23) of Lemma 2.2 with $\eta = 0$ and $\eta = 1$ we find:

[9]The key difference between this case and the pure-jump case actually centres on whether Y has infinite or finite variation; a similar result with the diffusion case below is also found for the pure jump model, if the jump process X has infinite variation.

Lemma 2.3. *In the case of the put boundary payoff* $\pi(y) = (K - e^y)_+ = K - e^y$*,*

$$\int_0^\infty \pi(a-x)\nu_x^*(\theta)dx \;=\; K\left(\frac{\nu^*(0)-\nu^*(\theta)}{\theta}\right) - L\left(\frac{\nu^*(1)-\nu^*(\theta)}{\theta-1}\right) \tag{24}$$

$$= -c(\theta)z(\theta) - \frac{\sigma^2 L}{2} - \frac{(r-q)L}{\theta-1} + \left(\frac{\sigma^2\theta}{2} + \mu\right)\pi(a). \tag{25}$$

where $z(\theta) = \frac{K}{\theta} - \frac{L}{\theta-1}$*, and* $L = e^a$ *(*z *is the transform of* $K - Le^u$*).*

Proof. The second expression (needed to establish 3.1b) is obtained by using the expression (5) connecting the functions $\nu_x^*(\theta), c(\theta)$ and the equalities $c(1) = r-q, c(0) = 0$.

The following lemma follows readily from elementary calculations which are left to the reader.

Lemma 2.4. *The Laplace transform in the surplus variable* u *of the final put payoff* $(K - Le^u)_+$ *is:*

$$f_a{}^*(\theta) \;=\; \frac{K}{\theta}(1-(L/K)^\theta) - \frac{L}{\theta-1}(1-(L/K)^{\theta-1})$$

$$= z(\theta) + \frac{K}{\theta(\theta-1)}(L/K)^\theta \tag{26}$$

where $L = e^a$ *and* $z(\theta) = \frac{K}{\theta} - \frac{L}{\theta-1}$

From Lemma 2.2, we obtain also a convenient form for the well known classical formula of the joint moment generating function $\Psi_{\delta,\eta}(u) = \mathbb{E}_u e^{-\delta\tau_0+\eta Y_{\tau_0}}$ –see [3], Theorem 6a, where τ_0 is the first time when 0 is overshot by a spectrally negative Lévy process.

Corollary 2.5. *The Laplace transform* $\Psi^*_{\delta,\eta}(\theta) = \int_0^\infty e^{-\theta u}\Psi_{\delta,\eta}(u)du$ *of the joint moment generating function* $\Psi_{\delta,\eta}(u) = \mathbb{E}_u e^{-\delta\tau_0+\eta Y_{\tau_0}}$ *is given by:*

$$\Psi^*_{\delta,\eta}(\theta) = (\delta - c(\theta))^{-1}\left(\frac{c(\theta)-c(\eta)}{\eta-\theta} - \frac{c(\delta_+)-c(\eta)}{\eta-\delta_+}\right) \tag{27}$$

Proof. By lemma 2.2 and (5), we get $\int_0^\infty \pi(-x)\nu_x^*(\theta)\,dx - \frac{\sigma^2}{2}\theta = \frac{c(\eta)-c(\theta)}{\theta-\eta} + \frac{\sigma^2}{2}\eta + \mu$. The result follows from Proposition 2.1(a) with $f = 0, a = 0, \pi(x) = e^{\eta x}$ and $\delta = \alpha + r$.

Note: It would be quite useful to extend somehow the approach of Proposition 2.1 to the case of Lévy processes with two sided jumps, which is the one of main interest in mathematical finance. In that case however we need to incorporate in the generator equation the contribution of the negative jumps as well, and taking Laplace transform ceases to yield an algebraic equation for the transform. The two-sided case requires considerably more work. For example, the joint moment generating function presented in Corollary 2.5 ceases to be a function of the Laplace exponent $c(\theta)$; instead (see [3] Theorem 1e) it requires obtaining the Wiener-Hopf factorization of the function $\frac{\delta}{\delta - c(\theta)}$, which is known explicitly only in a few particular cases.

3 The optimal barrier for Canadian American options

Proposition 2.1 yielded the over the barrier Laplace transform for the value $V_a(y)$ of a Canadian option. A further explicit formula for the value $V_a(y)$ itself will be provided in Proposition 4.3 below. By differentiating that, one might determine in principle the optimal barrier a^* for exercising a Canadian-American option. However, a more direct determination is obtained by using "smooth fit" continuity conditions for the values $V_a(0+), V_a'(0+)$ at the barrier (these were already obtained in (19), (20)).

More precisely, it is known that in the pure jump case the optimal value of a is given by the continuity condition $V_a(0+) = \pi(a)$ (see [8] or [13]) and in the diffusion case it is given by the smoothness condition $V_a'(0+) = \pi'(a)$ (see [8] or [12])).

We summarize in the Corollary below the results obtained from the smooth fit equations (parts b) and c) are needed to determine the optimal barriers of n-Erlangian and Canadian puts, and d) and e) are needed to establish the long/short term asymptotics of the approximate barrier).

Corollary 3.1. ***The optimal exercise barrier for Canadian American options***

a) The optimal exercise barrier a^* for a "stationary" option which solves the equation (15) must satisfy the nonlinear equation:

$$\alpha f_a^*(\delta_+) = -\int_0^\infty \pi(a-x)\nu_x^*(\delta_+)\, dx + (\frac{\sigma^2}{2}\delta_+ + \mu)\pi(a) + \frac{\sigma^2}{2}\pi'(a) \qquad (28)$$

b) For an arbitrary final payoff f and "put type" boundary payoff $\pi(y) = (K - e^y)_+$, *the optimal exercise boundary a satisfies:*

$$\alpha \left(f_a^*(\delta_+) - z(\delta_+) \right) = \frac{r\ K}{\delta_+} - \frac{q\ L}{\delta_+ - 1} \tag{29}$$

where $z(\theta) = \frac{K}{\theta} - \frac{L}{\theta - 1}$, *and* $L = e^a$.

c) In the case of both final and boundary payoff $f(y) = (K - e^y)_+$, *the optimal exercise boundary* $L = e^a$ *satisfies:*

$$\alpha \left(\frac{L}{K} \right)^{\delta_+} = (\delta_+ - 1) r - \delta_+ q \left(\frac{L}{K} \right) \tag{30}$$

d) In the particular case $\alpha = 0$ *we get*

$$\frac{L}{K} = \frac{r(\delta_r - 1)}{q \delta_r} = \frac{r(\delta_r - 1)}{\delta_r \big(c(\delta_r) - c(1) \big)} \tag{31}$$

where δ_r *is the positive root of the equation* $c(\theta) = r$. *If furthermore the dividend is* 0, *then* $\delta_r = 1$ *and the equation becomes:*

$$\frac{L}{K} = \frac{r}{c'(1)} = \frac{r}{\sigma^2 + \mu - \int_0^\infty x\, e^{-x}\, \nu(dx)}.$$

(This last formula has also been obtained in [8].)

e) In the limit $\alpha \to \infty$, *the solution of (30) becomes*

$$\frac{L}{K} = \min \left(\frac{r}{q}, 1 \right)$$

Remarks: 1) In the pure jump case (28) simplifies to:

$$\alpha f_a^*(\delta_+) = - \int_a^\infty \pi(a - x) \nu_x^*(\delta_+))\, dx + \mu \pi(a) \tag{32}$$

This generalizes the formula before the references in Zhang's(1998) discussion of Gerber and Shiu's paper [28].

2) Formula (30) generalizes formula (18) of Carr(1998) [7].

3) Since we plan to use (30) as an approximation for options with fixed finite expiry T, it is of interest to compare the approximations' limits as $T \to \infty$ and $T \to 0$ (obtained in parts d),e) by letting $\alpha \to 0$ and $\alpha \to \infty$ respectively) with the correct asymptotics of the exact optimal exercise barrier as $T \to \infty, T \to 0$. We see that at least in the Brownian

case (in which they are known) they do coincide with the well known exact asymptotics of the exercise barrier (see [18]). The limit as $T \to \infty$ (the $\alpha = 0$ case) in the pure Brownian case is usually written in the form

$$\frac{L}{K} = \frac{r}{r + \sigma^2 \delta_r / 2}$$

or in equivalent form

$$\frac{L}{K} = \frac{\tilde{\theta}_r}{\tilde{\theta}_r - 1}$$

where $\tilde{\theta}_r$ is the negative root of the quadratic equation $\sigma^2\theta^2/2 + (r - q - \sigma^2/2)\theta - r = 0$. We may easily check that these coincide in the Brownian case with our expression $\frac{r(\delta_r - 1)}{q\delta_r}$ provided in (31).

Proof of Corollary 3.1. a) The optimal exercise barrier is known to make the value function continuous at the barrier in the pure jump case $V_a(0+) = \pi(a)$, and to make the derivative of the value function continuous at the barrier in the jump diffusion case case $V_a'(0+) = \pi'(a)$ (see [8], [13] and [12]).

The equation $V_a'(0+) = \pi'(a)$ yields (28) after using the formula for $V_a'(0+)$ provided in Proposition c. In the case $\sigma = 0$, (28) becomes (32) which by Proposition 2.1(b) is tantamount to $V_a(0+) = \pi(a)$.

b) Using the second expression in Lemma 2.3 and the expression of the put's payoff $\pi(a) = K - e^a$, we rewrite (28) as:

$$\begin{aligned} \alpha f_a^*(\delta_+) &= \delta z(\delta_+) + \frac{\sigma^2 L}{2} + \frac{(r-q)L}{\delta_+ - 1} + \frac{\sigma^2 \pi'(a)}{2} \\ &= \alpha z(\delta_+) + \frac{rK}{\delta_+} + \frac{\sigma^2 L}{2} - \frac{qL}{\delta_+ - 1} - \frac{\sigma^2 L}{2} \\ &= \alpha z(\delta_+) + \frac{rK}{\delta_+} - \frac{qL}{\delta_+ - 1}, \end{aligned}$$

which establishes (29).

To obtain c) we substitute the expression $f_a^*(\theta) - z(\theta) = \frac{K}{\theta(\theta-1)}(L/K)^\theta$ provided in Lemma 2.4 for the transform of the put's final payoff into part b).

d) To obtain d) we note that the L.H.S. in (30) goes to 0 and that $q = c(\delta_r) - c(1)$.

e) It is known (see [8]) that $\lim_{\theta\to\infty} c(\theta)/\theta^2 = \sigma^2/2$, so that $\delta_+(\alpha) \sim O(\sqrt{\alpha})$ as $\alpha \to \infty$.

Dividing (30) by $q\delta_+$ and replacing δ_+ by its large α asymptotic, we get

$$q^{-1}\sqrt{\alpha}\exp\{\sqrt{\alpha}\log(L(\alpha)/K)\} + \exp\{\log(L(\alpha)/K)\} = \frac{(\sqrt{\alpha}-1)r}{q\sqrt{\alpha}} \to \frac{r}{q}.$$

Since $L(\alpha)/K \leq 1$, if $L(\alpha)/K \not\to 1$, the first term above converges to 0 while the second term is never bigger than 1. Therefore, if $r/q > 1$, the only way the left-hand side above can converge to a limit large than 1 is for the first term to converge to a non-zero limit, which implies $L(\alpha)/K \to 1$. Similarly, if $r/q \leq 1$, the first term above must converge to 0, so that $L(\alpha)/K \to r/q$. Therefore the $t \to 0$ asymptotic is

$$\frac{L}{K} = \min\left(\frac{r}{q}, 1\right)$$

which also agrees with the known asymptotic of the optimal boundary in the pure Brownian case. The same is also true in the pure jump case ($\sigma = 0$), only this time $\lim_{\theta\to\infty} c(\theta)/\theta = \mu$ so $\delta_+(\alpha) \sim O(\alpha)$. □

Corollary 3.2. *a) For an arbitrary final payoff f and "put type" boundary payoff $\pi(y) = (K - e^y)_+$, the Laplace transform in the surplus variable u of the value of a Canadian-American option $V_a(u)$ is given by:*

$$V^*(\theta) = \frac{b(\theta) - b(\delta_+)}{\delta - c(\theta)} \tag{33}$$

where $b(\theta) = \alpha f_a^*(\theta) - c(\theta)z(\theta) - (r-q)L/(\theta - 1)$, $L = e^a$ and $z(\theta) = \frac{K}{\theta} - \frac{L}{\theta-1}$.

b) After the barrier a^* is optimized, this becomes:

$$V^*(\theta) = z(\theta) + \frac{\alpha\left(f_{a^*}^*(\theta) - z(\theta)\right) - \left(\frac{rK}{\theta} - \frac{qL}{(\theta-1)}\right)}{\delta - c(\theta)} \tag{34}$$

Note: The expression above is well defined as $\theta \to \delta_+$ by the barrier optimality condition of Corollary 3.1b.

Proof. a) Substituting the expression (25) of the transform of the put's boundary payment into (18) yields (a) immediately.

b) The result follows upon noticing that

$$b(\delta_+) = \alpha f_a^*(\delta_+) - \delta z(\delta_+) - (r-q)L/(\delta_+ - 1) = \alpha(f_a^*(\delta_+) - z(\delta_+)) - rK/\delta_+ + qL/(\delta_+ - 1) = 0$$

(by the smooth fit condition (29) provided in Corollary 3.1b) and that $b(\theta)$ can also be written as

$$\begin{aligned} b(\theta) &= \alpha f_a^*(\theta) - c(\theta)z(\theta) - (r-q)L/(\theta-1) \\ &= z(\theta)(\delta - c(\theta)) + \alpha f_a^*(\theta) - \delta z(\theta) - (r-q)L/(\theta-1) \\ &= z(\theta)(\delta - c(\theta)) + \alpha(f_a^*(\theta) - z(\theta)) - rK/\theta + qL/(\theta-1). \quad \square \end{aligned}$$

4 The value of "Put type" Canadian barrier options

In Sections 2 and 3 we took advantage of the Laplace transform approach available for spectrally negative Lévy processes to obtain (in Proposition 2.1) a general formula for the Laplace transform in the initial capital for general Canadian barrier options. In this section we exploit yet another simplification of "fluctuation theory" available in the spectrally negative (positive) case: an explicit formula for the joint moment generating function of the time and place of exit from a one-sided interval in terms of the negative (positive) part of the resolvent density. Note that in the general case, the same task requires performing a Wiener-Hopf factorization – see Theorem 1e) of Bingham [3]. We quote now the spectrally one sided result, following Theorem 6b of Bingham [3].

Theorem 4.1. *Suppose that X is a Lévy process whose jumps are positive but which is not non-decreasing, with $X_0 = 0$.*

a) Then the resolvent measure $r_\delta(dx) = \int_0^\infty e^{-\delta t}\mathbb{P}(X_t \in dx)\,dt$ is absolutely continuous with respect to Lebesgue measure and the Laplace transform of its density $r_\delta(x)$, for $x > 0$ is given by

$$r_\delta^*(\theta) = \int_0^\infty e^{-\theta x} r_\delta(x)\,dx = \frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)}. \tag{35}$$

b) For $u > 0$, let $\tau_u = \inf\{t \geq 0 : \ X_t > u\}$. The joint law of τ_u and of the overshoot $X(\tau_u) - u$ is:

$$\Psi_{\delta,\eta}(u) = \mathbb{E}[e^{-\delta\tau_u - \eta(X(\tau_u)-u)}] = \big(\delta - c(\eta)\big)\left(\frac{r_\delta(u)}{\delta_+ - \eta} + \int_u^\infty e^{-\eta(z-u)} r_\delta(z)\,dz\right) \tag{36}$$

for $\delta, \eta \geq 0$.

Notes: 1) While the resolvent function is rarely available analytically, efficient numerical approaches for computing it exist, and so this provides an alternative to inverting the Laplace transform. For example, in [8] the Fast Fourier Transform algorithm is used to compute $r_\delta(x)$.

2) Relation (36) may be also obtained from Corollary 2.5. Indeed, starting from the end by taking Laplace transform in u of (36), we get:

$$\begin{aligned}\Psi^*_{\delta,\eta}(\theta) &= \big(\delta - c(\eta)\big)\left(\frac{r^*_\delta(\theta)}{\delta_+ - \eta} - \frac{r^*_\delta(\theta) - r^*_\delta(\eta)}{\theta - \eta}\right)\\ &= \frac{\delta - c(\eta)}{\delta_+ - \eta}\left(\frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)}\right)\\ &\quad - \frac{\delta - c(\eta)}{\theta - \eta}\left(\frac{1}{\delta - c(\theta)} - \frac{1}{c'(\delta_+)(\delta_+ - \theta)} - \frac{1}{\delta - c(\eta)} + \frac{1}{c'(\delta_+)(\delta_+ - \eta)}\right)\end{aligned}$$

in which the three terms involving $c'(\delta_+)$ cancel out and the remaining terms yield the result of Corollary 2.5 for the process $Y(t) = u - X(t)$.

3) The first relation (35) is a consequence of a simple identity

$$\frac{1}{\delta - c(\theta)} = r^*_\delta(\theta) + n^*_\delta(\theta) \tag{37}$$

where $n^*_\delta(\theta) = \int_{-\infty}^0 e^{-\theta u} r_\delta(u)\,du$ denotes the Laplace transform of the negative part of the resolvent density, and of the fact that the latter is the exponential function $r_\delta(u) = e^{\delta_+ u}/c'(\delta_+)$ (for $u < 0$) as shown for example in [3]. Let now $W(u)$ denote the inverse Laplace transform of the function $\frac{1}{\delta - c(\theta)}$. For general two sided Lévy processes, we have:

$$W(u) = \int_0^\infty e^{-\delta t}\mathbb{P}(X_t \in du)\,dt + \int_0^\infty e^{-\delta t}\mathbb{P}(X_t \in -du)\,dt \tag{38}$$

(by (37). We provide now an alternative expression for $W(u)$ available in the spectrally negative case:

Corollary 4.2. *The inverse Laplace transform of the function $\frac{1}{\delta - c(\theta)}$ for spectrally negative Lévy processes is given by*

$$W(u) = r_\delta(u) - \frac{e^{\delta_+ u}}{c'(\delta_+)} = \frac{\Psi_{\delta,\delta_+}(u) - e^{\delta_+ u}}{c'(\delta_+)} \tag{39}$$

Proof. The first equality is immediate from Theorem (4.1)a. The second follows by letting $\eta \to \delta_+$ in (36), which yields the resolvent density in terms of the joint Laplace transform of τ_u and $X(\tau_u)$:

$$r_\delta(u) = \frac{e^{\delta_+ u}}{c'(\delta_+)}\mathbb{E}[e^{-(\delta\tau_u + \delta_+ X(\tau_u))}] = \frac{\Psi_{\delta,\delta_+}(u)}{c'(\delta_+)}, \quad u > 0. \tag{40}$$

□

Note: Formula (40) shows that knowing the resolvent is equivalent to knowing the joint Laplace transform of the hitting time τ_u and of the overshoot $X(\tau_u)$, evaluated at the particular values which make the "Wald exponential" a martingale. In some special cases, like when X is a compound Poisson process with exponential jumps – see [13], or more generally, when the jumps have a phase-type distribution– see [1], the joint Laplace transform can be computed analytically, which makes (39) preferable to(38).

Using this corollary, we give now two explicit representations for the value of Canadian barrier options with put boundary payoff and arbitrary final payoff.

Proposition 4.3. *Let $L = e^a$ denote a fixed barrier and $u = y - a$ denote the initial log-surplus over the barrier a. If $X_t = y - Y_t$ satisfies the assumptions of Theorem 4.1, and $\pi(y) = (K - e^y)_+ = (K - Le^u)_+$ is the put barrier payoff, then*

$$V_{a,\pi,f,r,\alpha}(y) = V_a(u) = K\Psi_{\delta,0}(u) - L\Psi_{\delta,1}(u) + \int_0^u \alpha W(u-x) f_a(x)\, dx - \alpha W(u) f_a^*(\delta_+) \quad (41)$$

$$= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\, dz\right) - (\alpha + q)L\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty e^{u-z} r_\delta(z)\, dz\right)$$
$$+ \int_0^u \alpha\ r_\delta(u-x) f_a(x)\, dx - \alpha f_a^*(\delta_+) r_\delta(u) + \int_u^\infty \frac{e^{\delta_+(u-x)}}{c'(\delta_+)} f_a(x)\, dx. \quad (42)$$

Notes: 1) The first formula is more convenient for examples which allow an explicit Wiener-Hopf factorization, since both $\Psi_{\delta,\eta}(u)$ and $W(u)$ are expressible in terms of the Wiener-Hopf factors. The second formula is preferable if the resolvent is to be computed numerically.

2) Even though formula (41) makes perfect sense for spectrally two sided Lévy processes, we conjecture that it does **not** provide the value of Canadian American puts in that case.

Proof. It is enough to establish separately the two particular cases A: $f \equiv 0, \pi = (K - Le^u)_+$ (the payoff from overshooting the boundary) and B: $\pi \equiv 0$ (the payoff from expiration, if the boundary is not crossed).

Case A: When $f \equiv 0$, observe that the solution of (13) can be expressed probabilistically as

$$V_a(u) = \mathbb{E}_y[e^{-\delta\tau_u}\pi(Y(\tau_u))],$$

where the subscript in τ_u refers to the barrier of the process $X_t = y - Y_t$: $\tau_u = \inf\{t : X_t > u\} = \inf\{t : Y_t < a\}$. The function $V_a(u)$ may therefore be obtained directly by applying (36) to the spectrally positive process $X_t = y - Y_t$ with $\eta = 0$ and $\eta = 1$ and using $c(1) = r - q$, yielding

$$\begin{aligned}
V_a(u) &= \mathbb{E}[e^{-\delta\tau_u}(K - Le^{-(X(\tau_u)-u)})] = K\Psi_{\delta,0}(u) - L\Psi_{\delta,1}(u) \\
&= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\,dz\right) - L(\delta - c(1))\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty e^{-(z-u)} r_\delta(z)\,dz\right) \\
&= K\delta\left(\frac{r_\delta(u)}{\delta_+} + \int_u^\infty r_\delta(z)\,dz\right) - (\alpha + q)L\left(\frac{r_\delta(u)}{\delta_+ - 1} + \int_u^\infty e^{u-z} r_\delta(z)\,dz\right). \qquad (43)
\end{aligned}$$

Case B: When $\pi \equiv 0$, we recognize that the inverse Laplace transform of

$$\frac{\alpha(f_a^*(\theta) - f_a^*(\delta_+))}{\delta - c(\theta)}. \qquad (44)$$

is a combination of the inverse Laplace transform $W(u)$ of $\frac{1}{\delta - c(\theta)}$ provided in Corollary 4.2 and of the convolution of $W(u)$ with $f_a(x)$:

$$\int_0^u \alpha\left[r_\delta(u-x) - \frac{e^{\delta_+(u-x)}}{c'(\delta_+)}\right] f(x+a)\,dx - \alpha f_a^*(\delta_+)\left[r_\delta(u) - \frac{e^{\delta_+(u)}}{c'(\delta_+)}\right]. \qquad (45)$$

Finally, simplifying (45) and putting it together with (43) gives the desired result (42). $\square$

Note: The joint Laplace transform $\Psi_{\delta,\eta}$ may be computed explicitly for the compound Poisson model with phase-type jumps (see for example [1]), leading to an explicit formula for the resolvent density $r_\delta(u)$ via (40) In that case, (42) provides an explicit formula for the American put requiring only root solving and matrix exponentiation. Even simpler is the case of exponential jumps, presented in the Appendix.

5 The recursive algorithm

For a fixed positive integer n, let S_n be a random variable with $\Gamma(n, n/T)$ distribution, independent of Y. Since $\mathbb{E}(S_n) = T$ and $\mathrm{Var}(S_n) = T/n$, $S_n \to T$ in distribution as $n \to \infty$. The idea is to use

$$V_n(y) := \mathbb{E}_y[e^{-r(\tau_* \wedge S_n)}\pi(Y_{\tau_* \wedge S_n})]$$

for large n as an approximation to the price of an American put. Since $S_n = T_1 + T_2 + \cdots + T_n$ where T_i are independent exponential rate n/T random variables, the Markov property enables us to compute V_n using the following recursive algorithm:

$$\begin{aligned} V_0(y) &:= \pi(y) \\ V_i(y) &:= \mathbb{E}_y[e^{-r\tau_i}\pi(Y_{\tau_i})\mathbf{1}_{\{\tau_i \le T_i\}}] + \mathbb{E}_y[e^{-rT_i}V_{i-1}(Y_{T_i})\mathbf{1}_{\{\tau_i > T_i\}}], \quad i = 1, \dots, n \end{aligned} \tag{46}$$

where the stopping times τ_i are now the optimal exercise times of American type options with random expiry time T_i and final payoff given by V_{i-1}. The memoryless property of the exponential distribution means that the optimal exercise levels at each stage are now constants, so that the optimal exercise times τ_i are of form

$$\tau_i = \inf\{t \ge 0 : \; Y_t \le a_i\} \tag{47}$$

for some constant optimal levels a_i, which can be thought of as a piecewise constant approximation to the time-dependent optimal exercise curve for the fixed expiry option. Note that some care is needed when computing the final payoffs V_{i-1}, since these can come either from immediate stopping (in which case $V_{i-1}(y) = (K - e^y)_+$), or from continuing in which case $V_{i-1}(y)$ is given by Proposition 4.3.

With τ_i as in (47) and continuing with the previous notation $y = u + a_i$, consider the value of a put with the exercise level at each stage fixed at a_i:

$$V_i(u, a_i) = \mathbb{E}_{u+a_i}[e^{-r\tau_i}\pi(Y_{\tau_i})\mathbf{1}_{\{\tau_i \le T_i\}}] + \mathbb{E}_{u+a_i}[e^{-rT_i}V_{i-1}(Y_{T_i} - a_{i-1}, a_{i-1})\mathbf{1}_{\{\tau_i > T_i\}}]$$

and its Laplace transform

$$V_i^*(\theta, a_i) = \int_0^\infty e^{-\theta u} V_i(u, a_i)\, du.$$

(We already know that $V_i(x, a) = \Pi(a + x)$ for $x \le 0$, by definition.) Of course, the $V_i(u, a)$ and $V_i^*(u, a)$ are particular cases of the $V_a(u)$ and $V_a^*(u)$ considered in the previous section. At each stage we need to maximize $V_i(u, a)$ over a to obtain $V_i(y) = \sup_a V_i(y - a, a) = V_i(y - a_i, a_i)$. Note that the optimal exercise levels a_i do not depend on the starting point y (hence nor the initial excess u).

Before we can implement such a recursive algorithm, we need to verify that $\mathbb{E}[e^{-r(\tau_* \wedge S_n)} \pi(Y_{\tau_* \wedge S_n})]$ can indeed be used as an approximation for $\mathbb{E}[e^{-r(\tau_* \wedge T)}\pi(Y_{\tau_* \wedge T})]$ – more precisely, that

$$\mathbb{E}[e^{-r(\tau_* \wedge S_n)}\pi(Y_{\tau_* \wedge S_n})] \to \mathbb{E}[e^{-r(\tau_* \wedge T)}\pi(Y_{\tau_* \wedge T})]$$

as $n \to \infty$. Recall that $S_n \to T$ in distribution means that $\mathbb{E}[f(S_n)] \to \mathbb{E}[f(T)]$ for every bounded *continuous* function f. In the Brownian model considered in Carr [7], Y is continuous, so $\mathbb{E}[e^{-r(\tau_*\wedge S_n)}\pi(Y_{\tau_*\wedge S_n})] \to \mathbb{E}[e^{-r(\tau_*\wedge T)}\pi(Y_{\tau_*\wedge T})]$ essentially follows by definition. When Y is discontinuous, a little more work is required.

Lemma 5.1. *Let Y be a Lévy process and for fixed T let S_n be a sequence of random variables independent Y which converge to T in distribution. Then the sequence of processes $\{Y_{t\wedge S_n} : t \le T\}$ converge weakly to $\{Y_t : t \le T\}$.*

Proof. It is sufficient to prove that all the finite dimensional distributions of $\{Y_{t\wedge S_n} : t \le T\}$ converge to those of $\{Y_t : t \le T\}$. For simplicity, we first prove that $Y_{t\wedge S_n}$ converges in distribution to $Y_{t\wedge T}$ for each $t \le T$. Let f be a bounded continuous function, with $|f| \le K$. Let $\epsilon > 0$ be arbitrary. Choose δ_1 such that $|f(y_1) - f(y_2)| < \epsilon$ for all $|y_1 - y_2| < \delta_1$. Next, choose δ_2 such that whenever $|s-t| < \delta_2$, $\mathbb{P}|Y_s - Y_t| > \delta_1) \le 1 - e^{-\delta_2\nu(\delta_1,\infty)} < \epsilon$. Finally, choose n so large that $\mathbb{P}|S_n - T| > \delta_2) < \epsilon$. Then

$$\begin{aligned}
&|\mathbb{E}[f(Y_{t\wedge S_n})] - \mathbb{E}[f(Y_{t\wedge T})]| \\
&\le \mathbb{E}[|f(Y_{t\wedge S_n}) - f(Y_{t\wedge T}))|\mathbf{1}_{\{|S_n-T|>\delta_2\}}] \\
&\quad + \mathbb{E}[|f(Y_{t\wedge S_n}) - f(Y_{t\wedge T})|\mathbf{1}_{\{|S_n-T|\le\delta_2\}}\mathbf{1}_{\{|Y(t\wedge S_n)-Y(t\wedge T)|>\delta_1\}}] \\
&\quad + \mathbb{E}[|f(Y_{t\wedge S_n}) - f(Y_{t\wedge T})|\mathbf{1}_{\{|S_n-T|\le\delta_2\}}\mathbf{1}_{\{|Y(t\wedge S_n)-Y(t\wedge T)|\le\delta_1\}}] \\
&\le K\epsilon + K\epsilon + \epsilon.
\end{aligned}$$

In the same way, one can prove the convergence of all the finite dimensional distributions. $\square$

Corollary 5.2. *Let f be a continuous decreasing function. For any fixed $r > 0$,*

$$\sup_\tau \mathbb{E}[e^{-r(\tau\wedge S_n)}f(Y_{\tau\wedge S_n})] \to \sup_\tau \mathbb{E}[e^{-r(\tau\wedge t)}f(Y_{\tau\wedge t})]$$

(where the suprema are taken over all stopping times of the form $\tau = \tau(Y) = \inf\{t : Y_t \ge h(t)\}$ for some piecewise continuous curve $h(t)$).

Proof. It is clear from Lemma 5.1 that $\mathbb{E}[e^{-rS_n}f(Y_{S_n})] \to \mathbb{E}[e^{-rT}f(Y_T)]$. As functionals of the process $Y_{\cdot\wedge S_n}$, hitting times of the stated form are continuous in the topology of weak convergence and hence for each fixed stopping time τ, $\mathbb{E}[e^{-r(\tau\wedge S_n)}f(Y_{\tau\wedge S_n})] \to \mathbb{E}[e^{-r(\tau\wedge T)}f(Y_{\tau\wedge T})]$. But the stopping times $\tau\wedge S_n$ are bounded by $T+\epsilon$ on $\{|S_n - T| \le \epsilon\}$, an event whose probability can be made arbitrarily close to 1, so this convergence must be uniform in τ, that is, $\sup_\tau |\mathbb{E}[e^{-r(\tau\wedge S_n)}f(Y_{\tau\wedge S_n})] - \mathbb{E}[e^{-r(\tau\wedge T)}f(Y_{\tau\wedge T})]| \to 0$. The result now follows from the fact that

$$\left|\sup_\tau \mathbb{E}[e^{-r(\tau\wedge S_n)}f(Y_{\tau\wedge S_n})] - \sup_\tau \mathbb{E}[e^{-r(\tau\wedge T)}f(Y_{\tau\wedge T})]\right| \\ \leq \sup_\tau \left|\mathbb{E}[e^{-r(\tau\wedge S_n)}f(Y_{\tau\wedge S_n})] - \mathbb{E}[e^{-r(\tau\wedge T)}f(Y_{\tau\wedge T})]\right|. \quad \square$$

5.1 A recursive algorithm for the value of n-Erlang American puts

In the context of the recursive algorithm (46), at the i^{th} stage, we know the optimal exercise level a_{i-1} for the $(i-1)^{th}$ stage, $V_{i-1}(u, a_{i-1})$ and its Laplace transform $V^*_{i-1}(\theta, a_{i-1})$ (start with $a_0 = \log K$, $V_0(u) = \pi(u) = (K - e^u)_+$ and $V^*_0(\theta, a_0) = \int_0^{a_0} e^{-\theta u}\pi(u)\, du$). We take as the final payoff at the end of each stage i $f_i(z) = V_{i-1}(z - a_{i-1}, a_{i-1})$ (note that this depends on a_{i-1}), so that

$$f^*_i(\theta, a) = \int_0^\infty e^{-\theta u} V_{i-1}(u + a - a_{i-1}, a_{i-1})\, du.$$

But at the i^{th} stage we are looking for an exercise level $a \leq a_{i-1}$, so we have

$$f^*_i(\theta, a) = \int_0^{a_{i-1}-a} e^{-\theta u}\pi(u + a)\, du + e^{-\theta(a_{i-1}-a)} V^*_{i-1}(\theta, a_{i-1}). \tag{48}$$

To find the optimal exercise level a_i for the i^{th} stage and $V^*_i(\theta, a_i)$, we take $f^*(\theta, a)$ as given by (48) and solve (29) to find the optimal exercise level a_i, at which point (34) gives an expression for $V^*_i(\theta, a_i)$ in terms of quantities which are all known by the i^{th} stage. The formula (42) (with $f(z) = f_i(z) = V_{i-1}(z - a_{i-1}, a_{i-1})$ and $f^*_i(\theta, a_i)$ as given by (48)) gives an expression for $V_i(u, a_i)$. We now have $V^*_i(\theta, a_i)$ and $V_i(u, a_i)$ which are needed at the $(i+1)^{th}$ stage. Of course, in order to use (42), we need to compute the resolvent density $r_\delta(x)$ for $x > 0$. Although there is no explicit formula for this except in the case of only a few Lévy processes, it has been pointed out in [8] that $r_\delta(x)$ can be readily computed using the Fast Fourier Transform algorithm, at least when $\sigma \neq 0$; when $\sigma = 0$ the convergence of the relevant Fourier sums may be rather slow.

Alternatively, we can use a similar recursive algorithm to compute the Laplace transforms V^*_i at each stage (these are expressed solely in terms of V^*_{i-1} and π) and once V^*_n has been obtained, a single numerical Laplace inversion can be performed to obtain $V_n(u, a_n) = \mathbb{E}_{u+a_n}[e^{-r(\tau_*\wedge S_n)}\pi(Y_{\tau_*\wedge S_n})]$. However, inverting a Laplace transform numerically is usually more computationally demanding than the Fast Fourier Transform which is needed to compute $V_n(u, a_n)$ by means of Proposition 4.3. On the other hand, since Corollary 3.1 (especially part(b)) gives the optimal exercise level in terms of Laplace transforms f_i^* (and hence, by virtue of (48), V_{i-1}^*), it is possible to give more explicit formulae for the Laplace transforms V_i^* at each stage, at least when $\pi(a) = (K - e^a)_+$. Also, Proposition 4.3 only holds in the case $\pi(a) = (K - e^a)_+$, whereas (18) and (28) hold for general π.

5.2 An algorithm for the value of n-Erlang American puts via Laplace inversion

In this section, we present an algorithm for computing the value $V_n(u, a_n)$ of n-Erlang American puts via Laplace inversion. It must be emphasised that the algorithm does not need to invert a Laplace transform at each stage. Instead it computes the Laplace transform $V_n^*(\theta, a_n)$ explicitly and ends performing a single Laplace inversion.

In this section, we will denote by $L_k = e^{a_k}$ the successive barriers, by $f_k^* = f_{a_k}^*(\theta), V_k^* = V_k^*(\theta, a_k)$ the successive Laplace transforms of the final payoffs and value functions of the various stages and by $z_k(\theta)$ the expressions $\frac{K}{\theta} - \frac{L_k}{\theta-1}$.

Since Corollary 3.2b yields the Laplace transform $V_k^* - z_k$ in terms of $f_k^* - z_k$, and Corollary 3.1b expresses the optimal level L also in terms of $f_k^* - z_k$, it will be enough to obtain an explicit formula for $f_k^* - z_k$, and this is achieved in Proposition 5.3 below

Proposition 5.3. *Let $\pi(a) = (K - e^a)_+$. a) For $k \geq 2$, the transforms in u of the final payoff at the end of stage k satisfy the recursion:*

$$f_k^*(\theta) - z_k(\theta) \quad = \quad \left(\frac{L_k}{L_{k-1}}\right)^\theta \frac{\alpha}{\delta - c(\theta)} \left(f_{k-1}^*(\theta) - z_{k-1}(\theta) - \left(\frac{rK}{\alpha\theta} - \frac{qL_{k-1}}{\alpha(\theta-1)}\right)\right) \tag{49}$$

where L_k are defined recursively as solutions of:

$$\frac{rK}{\alpha\delta_+} - \frac{qL_k}{\alpha(\delta_+ - 1)} = \alpha \left(\frac{L_k}{L_{k-1}}\right)^{\delta_+} \lim_{\theta\to\delta_+} \frac{\left(f_{k-1}^*(\theta) - z_{k-1}(\theta) - \left(\frac{rK}{\alpha\theta} - \frac{qL_{k-1}}{\alpha(\theta-1)}\right)\right)'}{\delta - c'(\theta)} \tag{50}$$

b) The transform in u of the final payoff at the end of stage k is given by:

$$f_k^* - z_k = \left(\frac{\alpha}{\delta - c(\theta)}\right)^{k-1} \left(\frac{L_k}{K}\right)^\theta \frac{K}{\theta(\theta-1)} - \sum_{i=1}^{k-1} \left(\frac{\alpha}{\delta - c(\theta)}\right)^{k-i} \left(\frac{L_k}{L_i}\right)^\theta \left(\frac{rK}{\alpha\theta} - \frac{qL_i}{\alpha(\theta-1)}\right) \tag{51}$$

Notes: 1) The indeterminate limit in (50) yielding the optimal levels L_k requires using L'Hôpital's rule $k-1$ times. For example, for $k=2$ we find the optimality condition for the second barrier in the no dividends case to be:

$$\left(\frac{L_2}{L_1}\right)^{\delta_+} = \frac{c'(\delta_+)}{\alpha\left(\frac{1}{\delta_+-1} - \log\left(\frac{L_1}{K}\right)\right)}$$

2) It is possible to show using the smooth junction equation of Corollary 3.1b that the expressions (49), (51) are well defined as $\theta \to \delta_+$.

Proof. a) At the end of stage k, the barrier is switched from L_k to a higher value $L_{k-1} = L_k e^a$ where $a = \log(L_{k-1}/L_k)$. Thus, the payoff will either be an immediate exercise payoff $K - S_t$ if S_t is between the two barriers, or a continuation Canadian payoff given by (34). Thus, the Laplace transform at the k'th stage is:

$$\begin{aligned}
f_k^*(\theta) &= \int_0^a e^{-\theta u}(K - L_k e^u) + e^{-\theta a} V_{k-1}^*(\theta) \\
&= \frac{K}{\theta}\left((1 - \frac{L_k}{L_{k-1}})\right)^{\theta} - \frac{L_k}{\theta - 1}\left(1 - (\frac{L_k}{L_{k-1}})^{\theta-1}\right) + (\frac{L_k}{L_{k-1}})^{\theta} V_{k-1}^*(\theta) \\
&= z_k + \left(\frac{L_k}{L_{k-1}}\right)^{\theta} \left(V_{k-1}^* - z_{k-1}\right)
\end{aligned}$$

The recursion (49) follows now by plugging in $f_k^*(\theta) - z_k(\theta) = \left(\frac{L_k}{L_{k-1}}\right)^{\theta}\left(V_{k-1}^*(\theta) - z_{k-1}(\theta)\right)$ the formula for V_{k-1}^* provided in Corollary 3.2b.

The recursion (50) for the barriers is obtained from Corollary 3.1b by applying (49) and L'Hôpital's limit rule.

b) Follows by iterating the recursion of part a), starting with the expression $f_1 - z_1 = (\frac{L_1}{K})^{\theta} \frac{K}{\theta(\theta-1)}$ provided in Lemma 2.4.

□

6 Concluding remarks

As demonstrated by this paper, the value of American options with one exercise barrier may be approximated analytically under a spectrally one sided (negative or positive) exponential

Lévy model. It seems quite interesting to investigate whether this is still possible for the various spectrally two sided Lévy models recently proposed by Eberlein, Madan and Barndorf Nielsen, and for problems with two barriers.

Acknowledgement: We thank S. Asmussen and M. Pistorius for valuable comments.

Appendix: The value of Canadian Put options under the compound Poisson model with exponential jumps

In this section we assume that Y_t is compound Poisson with exponential jumps, i.e. we assume that $\sigma = 0$ and $J_t = \sum_{i=1}^{N_t} Z_i$, where N_t is a Poisson process with rate λ and Z_i have exponential density $\beta e^{-\beta x}$.

Let now $a = \log L$ and $k = \log K$ denote the exercise barrier and strike price of a put option on the logarithmic scale, with $a < k$. Since $V(y) = K - e^y$ for $y \leq a$, we find that the differential equation for the value of Canadian options (13) becomes in the put case:

$$
\begin{aligned}
&(GV)(y) - \delta V(y) + \lambda \int_{y-a}^{\infty} \big(K - e^{y-z}\big) p(z) dz + \alpha (K - e^y)_+ = \\
&(GV)(y) - \delta V(y) + \lambda \big(K \bar{P}(y-a) - L p^*_{y-a}(1)\big) + \alpha (K - e^y)_+ = 0 \qquad \text{for } y \geq a \qquad (52)\\
&V(y) = K - e^y \qquad \text{for } y \leq a
\end{aligned}
$$

where $(GV)(y) = \mu V'(y) + \lambda \int_0^{y-a} [V(y-z)]\, p(z) dz - \lambda V(y)$ and V is differentiable over (a, k) and (k, ∞) and continuous at k.

In the exponential case we have $p^*_u(\theta) = \frac{p(u)}{\theta + \beta}$, and so (52) becomes:

$$
\begin{aligned}
&(\bar{G}\bar{V})(y) - \delta \bar{V}(y) + \lambda \int_{y-k}^{y-a} V(y-z) p(z) dz + \lambda p(y-a) \Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big) = 0 \quad \text{for } y \geq k \quad (53)\\
&(GV)(y) - \delta V(y) + \lambda p(y-a) \Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big) + \alpha (K - e^y) = 0 \quad \text{for } a \leq y \leq k \quad (54)\\
&\tilde{V}(y) = K - e^y \qquad \text{for } y \leq a
\end{aligned}
$$

where $\bar{V}, V, \tilde{V}$ denote the value of the option in the three ranges and $\bar{G}$ is the operator $(\bar{G}f)(y) = \mu f'(y) + \lambda \int_0^{y-k} [f(y-z)]\, p(z)dz - \lambda f(y)$

Using the results of the previous section, we may check that the solution of (52) is a combination of exponential functions. However, once the right form of the solution is known it becomes simpler to determine it directly, using the easily checked:

Lemma A.1. *The operator G acts on exponential functions as:*

$$Ge^{\theta y} = c(\theta)e^{\theta y} - e^{\theta a}\lambda p^*_{y-a}(\theta) \tag{55}$$

The equation $c(\theta) = \mu\theta + \frac{\lambda\beta}{\theta+\beta} - \lambda = \mu\theta - \frac{\lambda\theta}{\theta+\beta} = \delta$ becomes in this case: $\mu\theta^2 + \theta(\mu\beta - (\lambda+\delta)) - \delta\beta$. Let δ_+, δ_- denote the positive and negative roots of this equation[10].

The structure of the equations (53) forces us to start with the middle case $a \le y \le k$, for which we guess a solution of the form $V(y) = k_1 e^{\theta_1 y} + k_2 e^{\theta_1 y} + A_1 + A_2 e^y$ (the two undetermined exponents end up being the two roots of the equation $c(\theta) = \delta$). Plugging in the equation (54) and using the fact that

$$Ge^{\theta y} = e^{\theta y}c(\theta) - \frac{\lambda p(y-a)e^{\theta a}}{\beta+\theta} \tag{56}$$

yields

$$k_1\left(e^{\theta_1 y}(c(\theta_1)-\delta) - \frac{\lambda p(y-a)e^{\theta_1 a}}{\beta+\theta_1}\right) + k_2\left(e^{\theta_2 y}(c(\theta_2)-\delta) - \frac{\lambda p(y-a)e^{\theta_2 a}}{\beta+\theta_2}\right) +$$
$$A_1\left((c(0)-\delta) - \frac{\lambda p(y-a)}{\beta}\right) + A_2\left(e^y(c(1)-\delta) - \frac{\lambda p(y-a)e^a}{\beta+1}\right) +$$
$$\lambda p(y-a)\Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big) + \alpha(K - e^y) = 0$$

Using $c(1) = r - q$ and $c(0) = 0$ and putting $\theta_1 = \delta_+, \theta_2 = \delta_-$ we find that $A_1 = K\frac{\alpha}{\alpha+r}$ and $A_2 = -\frac{\alpha}{\alpha+q}$ and k_1, k_2 satisfy:

$$\frac{k_1 e^{\delta_+ a}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- a}}{\beta+\delta_-} = \Big(\frac{Kr}{\beta(\alpha+r)} - \frac{Lq}{(\beta+1)(\alpha+q)}\Big).$$

For the case $y \ge k$, we guess a solution of the form $\bar{V}(y) = \bar{k}e^{\theta y}$ with $\theta = \delta_-$ (since this domain is unbounded and the solution has to be bounded as $y \to \infty$, θ must be negative).

[10] Including the diffusion term leads to a similar analysis, the only difference being that the equation $c(\theta) - \delta$ will be of third order

Substituting this into (53) and using (56) and $\int_{y-k}^{y-a} Ce^{\theta(y-z)}\beta e^{-\beta z}dz = \frac{C}{\beta+\theta}\big(e^{\theta k}p(y-k) - e^{\theta a}p(y-a)\big)$ yields

$$\bar{k}\left(e^{\theta y}(c(\theta)-\delta) - \frac{\lambda p(y-k)e^{\theta k}}{\beta+\theta}\right) + \lambda\int_{y-k}^{y-a} V(y-z)p(z)dz + \lambda p(y-a)\Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big)$$

$$= -\bar{k}\frac{\lambda p(y-k)e^{\delta_- k}}{\beta+\delta_-} + \lambda\beta\int_{y-k}^{y-a}(k_1 e^{\delta_+ y-(\delta_+ +\beta)z} + (k_2 e^{\delta_- y-(\delta_- +\beta)z} + A_1 e^{-\beta z} + A_2 e^{y-(\beta+1)z})\,dz$$
$$+ \lambda p(y-a)\Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big)$$

$$= \lambda\Big\{ - \bar{k}\frac{p(y-k)e^{\delta_- k}}{\beta+\delta_-} + \frac{k_1}{\delta_+ +\beta}\big(p(y-k)e^{\delta_+ k} - p(y-a)e^{\delta_+ a}\big)$$
$$+ \frac{k_2}{\delta_- +\beta}\big(p(y-k)e^{\delta_- k} - p(y-a)e^{\delta_- a}\big) + \frac{A_1}{\beta}\big(p(y-k) - p(y-a)\big)$$
$$+ \frac{A_2}{\beta+1}\big(p(y-k)e^{k} - p(y-a)e^{a}\big) + p(y-a)\Big(\frac{K}{\beta} - \frac{L}{\beta+1}\Big)\Big\} = 0.$$

The constants k_1 and k_2 have been chosen so that all the terms involving $p(y-a)$ in the above expression cancel. We must therefore choose $\bar{k}$ so that

$$\frac{\bar{k}e^{\delta_- k}}{\beta+\delta_-} = \frac{k_1 e^{\delta_+ k}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}}{\beta+\delta_-} + \frac{A_1}{\beta} + \frac{A_2 e^{k}}{\beta+1} = \frac{k_1 e^{\delta_+ k}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}}{\beta+\delta_-} + \frac{K\alpha}{\beta(\alpha+r)} - \frac{K\alpha}{(\beta+1)(\alpha+q)}$$

Finally, using the continuity at k we find:

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} + \frac{K\alpha(q-r)}{(\alpha+r)(\alpha+q)} = \bar{k}e^{\delta_- k}$$

We have now proved the following result.

Proposition A.2. *The value of a Canadian put option on a compound Poisson process with exponential jumps is given by:*

$$V(y) = \begin{cases} k_1 e^{\delta_+ y} + k_2 e^{\delta_- y} + \frac{K\alpha}{\alpha+r} - \frac{\alpha}{\alpha+q}e^{y} & \text{for } a \le y \le k \\ \bar{k}e^{\delta_- y} & \text{for } y \ge k \end{cases} \tag{57}$$

where

$$\frac{k_1 e^{\delta_+ a}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- a}}{\beta+\delta_-} = \Big(\frac{Kr}{\beta(\alpha+r)} - \frac{Lq}{(\beta+1)(\alpha+q)}\Big)$$

$$\frac{k_1 e^{\delta_+ k}}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}}{\beta+\delta_-} = \frac{\bar{k}e^{\delta_- k}}{\beta+\delta_-} - \frac{K\alpha}{\beta(\alpha+r)} + \frac{K\alpha}{(\beta+1)(\alpha+q)}$$
$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k}e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}.$$

We compare now the results for exponential jumps with those obtained by approximating our Lévy process by a Brownian motion of equal variance $\sigma^2 = 2\lambda\beta^{-2}$ (and $\mu = \frac{\sigma^2}{2}\left(\frac{\beta}{\beta+1}\right)^2(\beta+1)+r-q$), where $\beta \to \infty$. In the limit, δ_+, δ_- become the positive and negative root of the quadratic equation $\frac{\sigma^2}{2}\theta^2 + (r-q-\frac{\sigma^2}{2})\theta - (r+\alpha) = 0$. We multiply the first equation by β and replace the second equation in the system above by the third minus β times the second:

$$\frac{k_1 e^{\delta_+ a}\beta}{\beta+\delta_+} + \frac{k_2 e^{\delta_- a}\beta}{\beta+\delta_-} = \Big(\frac{Kr}{(\alpha+r)} - \frac{Lq\beta}{(\beta+1)(\alpha+q)}\Big)$$

$$\frac{k_1 e^{\delta_+ k}\delta_+}{\beta+\delta_+} + \frac{k_2 e^{\delta_- k}\delta_-}{\beta+\delta_-} = \frac{\bar{k} e^{\delta_- k}\delta_-}{\beta+\delta_-} + \frac{K\alpha}{(\beta+1)(\alpha+q)}$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}.$$

Finally, multiplying the second equation by β and taking the limit $\beta \to \infty$ yields:

$$k_1 e^{\delta_+ a} + k_2 e^{\delta_- a} = \Big(\frac{Kr}{(\alpha+r)} - \frac{Lq}{\alpha+q}\Big)$$

$$k_1 e^{\delta_+ k}\delta_+ + k_2 e^{\delta_- k}\delta_- = \bar{k} e^{\delta_- k}\delta_- + \frac{K\alpha}{\alpha+q}$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{K\alpha(r-q)}{(\alpha+r)(\alpha+q)}.$$

If $q = 0$, this becomes:

$$k_1 e^{\delta_+ a} + k_2 e^{\delta_- a} = \frac{Kr}{\alpha+r}$$

$$k_1 \delta_+ e^{\delta_+ k} + k_2 \delta_- e^{\delta_- k} - K = \bar{k}\delta_- e^{\delta_- k}$$

$$k_1 e^{\delta_+ k} + k_2 e^{\delta_- k} = \bar{k} e^{\delta_- k} + \frac{Kr}{\alpha+r}.$$

which agrees with the results of Carr [7] when the boundary L is optimized.

References

[1] S. Asmussen, F. Avram, and M. Usabel (2000), "Finite time multivariate ruin distribution with one Laplace inversion", Preprint.

[2] O. E. Barndorff-Nielsen and N. Sheppard (2000). "Modelling by Lévy processes for financial econometrics." In Lévy processes, eds. O.E. Barndorff-Nielsen, T. Mikosh and S. Resnick, Birkhauser, Boston.

[3] N. H. Bingham (1975), "Fluctuation theory in continuous time", *Advances in Applied Probability*, **7**, 705 – 766.

[4] S. I. Boyarchenko and S. Z. Levendorskii(2000). "Perpetual American options under Lévy processes". Preprint.

[5] M. Broadie and J. Detemple (1996), "American option valuation: new bounds and approximations", *Review of Financial Studies*, **9**, 1211 – 1250.

[6] P. Carr, R. Jarrow, R. Myneni (1992), "Alternative characterizations of American put options", *Mathematical Finance*, **2**, 87 – 105.

[7] P. Carr (1998), "Randomization and the American put", *Review of Financial Studies*, **11**, 597 – 626.

[8] T. Chan (1999), "Pricing perpetual American options driven by spectrally one-sided Lévy processes", preprint.

[9] E. Eberlein and U. Keller (1995) "Hyperbolic Distributions in Finance". *Bernoulli* **1**, 281–299.

[10] H. Geman, D. Madan and M. Yor (1999) "Asset Prices Are Brownian Motion: Only in Business Time". Preprint, University of Maryland.

[11] H. Gerber and E. Shiu (1994). "Option Pricing by Esscher transforms". *Trans. Soc. Actuaries* **XLVI**, 99–139.

[12] H. Gerber and B. Landry (1998), "On the discounted penalty at ruin in a jump-diffusion and the perpetual put option", *Insurance: Maths & Econ*, **22**, 263 – 276.

[13] H. Gerber and E. Shiu (1998), "Pricing perpetual options for jump processes", *North Amer. Act. Journal*, **2**, 101 – 112.

[14] J. Kim (1990), "The analytic valuation of American puts", *Rev. Financial Studies* **3**, 547–572.

[15] S. D. Jacka(1991), "Optimal Stopping and the American put", *Math. Finance* **1**, 1–14.

[16] N. Ju (1998) "Pricing an American option by approximating its exercise boundary as a multipiece exponential", *Review of Financial Studies* **11**, pp 627-646.

[17] N. Ju and R. Zhong (1999) "An approximate formula for pricing American options", *J. Derivatives*, **7**, 31 – 40.

[18] Y. K. Kwok (1998) "Mathematical models of financial derivatives", *Springer Verlag.*

[19] D. Madan (1999) "Purely discontinuous asset price processes", Preprint, University of Maryland.

[20] H. P. McKean and P. A. Samuelson (1965), "A free boundary problem arising in mathematical economics", *Ind. Management Rev.*, **6**, 32 – 39.

[21] L. McMillan (1986), "Analytic approximation for the American put option", *Advances in futures and options research* **1**, 119–140.

[22] E. Omberg (1987), "The valuation of American put options", *Advances in futures and options research* **1**, 117-142.

[23] H. Pham (1997), "Optimal stopping, free boundary and American option in a jump-diffusion model",*Appl. Math. Opt.*,**35**, 145 – 164.

[24] D. Revuz and M. Yor (1991). "Continuous Martingales and Brownian motion", Springer.

[25] M. Usabel (1999), "Multivariate finite time ruin probabilities for hyper-exponential waiting times", Documento de Trabajo. Universidad Carlos III de Madrid.

[26] X. L. Zhang (1995), "Formules quasi-explicites pour les options américaines dans un modèle de diffusion avec sauts", *Math. and computers in simulation* 38, 151-161.

[27] X. L. Zhang (1997), "Numerical analysis of American option pricing in a jump-diffusion model", *Math of Oper. Res.* 22, 3, 668-690.

[28] X. L. Zhang (1998), Discussion to H. Gerber and E. Shiu (1998), "Pricing perpetual options for jump processes", *North Amer. Act. Journal*, **2**, 101 – 112.