Ruin probabilities and deficit for the renewal risk model with phase–type interarrival times

F. Avram [1] and M. Usabel[2]
[1] Dept. of Mathematics, Université de Pau, France
[2] Dept. of Business Adm., Universidad Carlos III de Madrid, Spain

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Abstract We provide below a generalization of Thorin's formula(1971) for the **double Laplace transform** of the finite time ruin probability, by considering also the deficit at ruin; the model is that of a Sparre Andersen (renewal) risk process with **phase–type interarrival times**.
In the case when the claims distribution is of phase–type as well, we obtain also an alternative formula for the **single Laplace transform in time** (or "exponentially killed probability"), in terms of the roots with **positive real part** of the Lundberg's equations, which complements Asmussen's representation(1992) in terms of the roots with negative real part.

1. Introduction

The convenience of phase–type modelling has been amply demonstrated in both risk and queueing theory; for example, the assumption of phase–type claims leads to an analytic formula for the **single Laplace transform in time** of the finite ruin probability (see Thorin(1971), Asmussen(1992), Asmussen, Avram and Usábel(2001) and Avram and Usábel(2001, 2002). A further illustration of this is provided below; under the assumption of phase–type interarrival times, we extend the formula of the double Laplace transform of the finite time ruin probability of a renewal risk process, obtained by Thorin(1971) and by Cohen(1985) in the context of queueing theory, to include the deficit at ruin as well. Our solution emphasizes the importance of a certain "structure matrix" (see below), already encountered in the previously studied case of phase–type claims.

We consider a Sparre Andersen risk process in continuous time $\{Z_t\}_{t\geq 0}$ with U_k claim sizes and premium c per time unit, i.e.

$$Z_t = u + ct - \sum_{k=1}^{N_t} U_k \tag{1.1}$$

where u are the initial reserves and N_t the total number of claims up to time t. The claim times sequence forms an ordinary renewal process with common c.d.f. of the waiting times between claims $A(t)$ with mean $1/\lambda$. The claims $\{U_k\}$ are independent one another and also independent of N_t, with c.d.f. $B(u)$ the distribution function of claim sizes U_k with mean μ. The distributions $A(t), B(u)$ are concentrated on $(0, \infty)$ and have densities and Laplace transforms denoted by $a(t), b(u)$ and $a^*(m), b^*(n)$, respectively. We consider here the case when furthemore $A(t)$ and sometimes $B(u)$ are of "phase–type", i.e. matrix exponential distributions (see below).

Let us now define $\tau = \inf\{w > 0 : Z_w < 0\}$ as the ruin time and $Y = -Z_\tau$ as the deficit exactly at ruin time or severity of ruin. The probability of ruin within time span t, initial reserves u and severity of ruin less than y is defined as:

$$P\{\tau < t, Y \leq y\} = \Psi_{t,u,y}$$

(the particular case $\Psi_{u,y} = P\{\tau < \infty, Y \leq y\}$ being known as the perpetual ruin probability).

The density function of the ruin time

$$\psi_{t,u,y} = \frac{\partial\, \Psi_{t,u,y}}{\partial t}$$

has been also treated frequently in actuarial and queueing literature; see the related works by Egidio dos Reis(1993,2000) and Dickson and Egidio dos Reis(1996) and Willmot(1998,2000).

Using a renewal argument, see for instance Thorin(70,71,82), we can easily obtain the following integral equation for the multivariate ruin probability(with deficit at ruin y) for the ordinary case

$$\begin{aligned}\Psi_{t,u,y} = & \int_0^t (B(u+cs+y) - B(u+cs))\ a(s)\ ds \\ & + \int_0^t \int_0^{u+cs} \Psi_{t-s,\ u+cs-z,\ y}\ b(z)\ a(s)\ dz\ ds\end{aligned}$$

The corresponding equation for the density is:

$$\begin{aligned}\psi_{t,u,y} = & \ (B(u+ct+y) - B(u+ct))\ \ a(t) \\ & + \int_0^t \int_0^{u+cs} \psi_{t-s,\ u+cs-z,\ y}\ b(z)\ a(s)\ dz\ ds \qquad (1.2)\end{aligned}$$

The classical approach to the ruin problem, going back to Cràmer(1955) and Sparre Andersen(1955), consists in applying integral transforms, like for example the double Laplace-Stieltjes operator:

$$\Psi^{**}_{m,n,y} = \int_{t=0}^{\infty} \int_{u=0}^{\infty} \Psi^{**}_{t,u,y}\ e^{-mt-nu}\ dudt$$

to the integral equations above (of course, the two results are related via $\psi^{**}_{m,n,y} = m\,\Psi^{**}_{m,n,y}$). See Thorin (1970,71,73,77,82), Thorin and Wikstad(1973), Wikstad(1971,77) who used the Piessens(1969) inversion method of the Laplace transform, Bohman(1971,74,75) who used the Fourier transform and Seal(1971,74,77) who dealt with both Laplace and Fourier numerical inversions.

In the classical Levy case with exponential interarrival times (see Cràmer(1955)), this approach culminated in the "Wiener-Hopf factorization", which yields in particular the double Laplace transform of the ruin probability. A further analytic expression for the double transform $\psi^{**}(n,m)$ was later obtained by Thorin(1971) for the more general renewal model. Below, we generalize Thorin's result to include the ruin at deficit, under the assumption that the interarrival distribution is of *phase–type*.

Definition 1. *A* **phase–type** *distribution $B(x)$ with parameters $(\boldsymbol{\beta}, \boldsymbol{B}, k)$ is the distribution of the absorbtion time ξ of a finite (transient) Markov process J_t on a state space $\{C, 1, \ldots, k\}$ with $k+1$ states to an absorbing state C called cemetery. Here, $\boldsymbol{B}$ denotes the restriction of the intensity matrix of J to the transient states $\{1, \ldots, k\}$, $\boldsymbol{\beta}$ denotes the initial distribution (written as a row vector) of J_0 on $\{1, \ldots, k\}$, and $\xi = \inf\{t > 0 : \ J_t = C\}$*

It follows that

$$
\begin{aligned}
B(x) &= \mathbb{P}_{\boldsymbol{\beta}}\left[\xi \le x\right] = 1 - \boldsymbol{\beta}\, e^{\mathbf{B}x}\,\mathbf{1} \\
b(x) &= \boldsymbol{\beta} e^{\boldsymbol{B}x} \boldsymbol{b}
\end{aligned}
$$

where $\mathbf{1} = (1 \ldots 1)'$, $\boldsymbol{b} = -\boldsymbol{B}\mathbf{1}$, and the Laplace transform of the density is:

$$
\begin{aligned}
b^*(s) &= \boldsymbol{\beta}\ (s\mathbf{I} - \mathbf{B})^{-1}\ \mathbf{b} \\
&= \boldsymbol{\beta}\, \boldsymbol{b}^*[s] \ \ \text{(see item 2 below)}
\end{aligned}
$$

Notes: 1) The vector $\boldsymbol{b}$ yields the rates of absorbtion from the various phases.

2) The column vector of Laplace transforms $\mathbf{b}^*[s] = (b_i^*(s),\ i = 1, \ldots, k)$, where $b_i^*(s) = \mathbb{E}_i\, e^{-s\xi}$ denotes the Laplace transform of the absorbtion time starting in phase i, given by

$$
\mathbf{b}^*[s] = (s\mathbf{I} - \mathbf{B})^{-1}\ \mathbf{b}
$$

3) Since the RLT/phase–type distributions are dense in the family of all distributions, this approach is of considerable practical interest.

4) The phase-type distributions have been used frequently in the recent actuarial and queueing literature. The reader is referred to the works by Neuts

(1975,1977), Asmussen(1987,92), Asmussen and Rolski(1991) and Asmussen and Bladt(1996) for further properties and applications.

Assumption: We assume throughout the paper that the interarrival distribution is of phase type $(\boldsymbol{\alpha}, \boldsymbol{A}, r)$, i.e. that

$$A(t) \;=\; 1 - \boldsymbol{\alpha} e^{t\boldsymbol{A}} \mathbf{1}$$

where $\mathbf{A}$ is of order r and nonsingular.

Example 1: The *hyperexponential* distribution used by Thorin(1971) is a convex combination of exponentials. The density

$$a(t) = \sum_{i=1}^{r} \alpha_i \nu_i e^{-\nu_i t}, \text{ where } \nu_i > 0, \alpha_i > 0, \sum_{i=1}^{r} \alpha_i = 1 \tag{1.3}$$

is $(\boldsymbol{\alpha}, \boldsymbol{A}, r)$ phase–type with $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 & \cdots & \alpha_r \end{pmatrix}$ and $\boldsymbol{A} = Diag\,\{\nu_i\}$. In this case, the vector of Laplace transforms is

$$\mathbf{a}^*[n] = (n\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{a} = \begin{pmatrix} \frac{\nu_1}{\nu_i+n} \\ \vdots \\ \frac{\nu_r}{\nu_r+n} \end{pmatrix}$$

$$\mathbf{a} = -\mathbf{A}\,\mathbf{1}$$

.

Example 2: The distribution of an *Erlang*(p) random variable is phase-type $(\boldsymbol{\alpha}, \boldsymbol{A}, p)$ with

$$\boldsymbol{A} \;=\; \begin{pmatrix} -\nu & \nu & 0 & 0 \\ 0 & -\nu & \nu & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & -\nu & \nu \\ 0 & 0 & 0 & \nu \end{pmatrix} \tag{1.4}$$

and $\boldsymbol{\alpha} = (1, 0, 0, \ldots.0)$ (the density is $a(t) = \frac{\nu^p}{(p-1)!}\, t^{p-1} e^{-\nu\, t}, \quad t > 0$).

For a general $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_p)$ where $\alpha_i > 0$, $\sum_{i=1}^{p} \alpha_i = 1$ we obtain a *Coxian* $Cox(p, \nu, \boldsymbol{\alpha})$ random variable with density

$$a(t) = \sum_{i=1}^{p} \alpha_i \,\frac{\nu^{\,i}}{(i-1)!}\, t^{i-1} e^{-\nu\, t}$$

and in both cases

$$\mathbf{a}^*[n] = (n\,\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{a} = \begin{pmatrix} \left(\frac{\nu}{\nu+n}\right)^{p+1-1} \\ \vdots \\ \left(\frac{\nu}{\nu+n}\right)^{p+1-p} \end{pmatrix}$$

It is easy to see that any distribution $F(x)$ concentrated on $(0,\infty)$ may be approximated arbitrarily close by Coxians $Cox(k^2+1,k,\boldsymbol{\alpha})$ with k a large integer, by choosing $\alpha_i = F(\frac{i}{k}) - F(\frac{i-1}{k}), i = 1, ..., k^2$ and $\alpha_{k^2+1} = 1 - F(k)$.

Example 3: *Hypercoxian* distributions are convex combinations of Coxians with p.d.f:

$$a(t) = \sum_{k=1}^{l} \sum_{i=1}^{p_k} \alpha_{k,i} \frac{\nu_k^i}{(i-1)!} t^{i-1} e^{-\nu_k t}, \quad t > 0; \tag{1.5}$$
$$\sum_{i,j} \alpha_{k,i} = 1$$

with ν_k positive and distinct and p_k integers, for k = 1, ..., l. The Laplace transform is:

$$a^*(n) = \sum_{k=1}^{l} \sum_{i=1}^{p_k} \alpha_{k,i} \left(\frac{\nu_k}{n+\nu_k}\right)^i \tag{1.6}$$

and so this class coincides with the set of distributions of nonnegative random variables with rational Laplace transform and only real (negative) poles. The phase-type representation $(\boldsymbol{\alpha}, \boldsymbol{A}, p)$ can be given, in block notation, by

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\alpha}_1 & \cdots & \boldsymbol{\alpha}_l \end{pmatrix}; \; \boldsymbol{A} = diag\left\{\mathbf{A}_k\right\}; \;\; p = \sum_{k=1}^{l} p_k$$

because

$$e^{diag\{\mathbf{A}_k\}} = diag\left\{e^{\mathbf{A}_k}\right\}$$

where for $k = 1, ..., l$

$$\boldsymbol{\alpha}_k = \begin{pmatrix} \alpha_{k,1} & \cdots & \alpha_{k,p_k} \end{pmatrix}; \;\; \mathbf{A}_k = \begin{pmatrix} -\nu_k & \nu_k & 0 & 0 \\ 0 & -\nu_k & \nu_k & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & -\nu_k & \nu_k \\ 0 & 0 & 0 & \nu_k \end{pmatrix}$$

The vector of Laplace transforms $\mathbf{a}^*[n]$ has then a block structure:

$$\mathbf{a}^*[n] = \begin{pmatrix} (n\,\mathbf{I}_1 - \mathbf{A}_1)^{-1}\,\mathbf{a}_1 \\ \vdots \\ (n\,\mathbf{I}_l - \mathbf{A}_l)^{-1}\,\mathbf{a}_l \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^*\,[n] \\ \vdots \\ \mathbf{a}_l^*\,[n] \end{pmatrix}$$

with each block of the form from the previous example for $k = 1, ..., l$

$$\mathbf{a}_k^*[n] = \begin{pmatrix} \left(\frac{\nu_k}{\nu_k+n}\right)^{p_k+1-1} \\ \vdots \\ \left(\frac{\nu_k}{\nu_k+n}\right)^{p_k+1-p_k} \end{pmatrix} \tag{1.7}$$

Contents: In the present work we obtain in Theorem 1 the double Laplace transform for the multivariate problem (considering the deficit at ruin y) under the assumption of phase–type interarrivals, generalizing formula 3.4 of Thorin(1971).

In section 3, corollary 2, an application of theorem 1 is presented regarding the calculation of the exponentially killed multivariate ruin probability for phase-type claim sizes. This result, expressed in terms of the non-negative roots (not necessarily distinct) of the Lundberg's equation generalizes several particular cases which have been presented in the literature like Dickson and Hipp(1998,2000) and Willmot(1999) and yields an alternative representation of Asmussen's result(1992) which had used the negative roots.

Note: As discussed in Asmussen, Avram and Usábel(2001), we may strengthen this result to incoporate also phase–type positive jumps and Brownian motion in the risk model, by adopting a semi-Markov embedding approach. The point of the curent paper on the other hand is to provide direct proofs in a simple but still quite general context, by taking advantage of the straightforward computation of the Laplace transform.

2. The double Laplace transform of the finite time multivariate ruin probability

Theorem 1. *a) If the waiting times of a risk process are distributed following a phase-type distribution* $(\boldsymbol{\alpha}, \mathbf{A}, r)$ *with p.d.f.* $a(t) = \boldsymbol{\alpha}\, e^{\mathbf{A}\,\mathbf{t}}\,\mathbf{a}$ *and Laplace transform,* $a^*(n) = \boldsymbol{\alpha}\ \mathbf{a}^*\,[n\,]$, *then the double Laplace transform of the density function of the time to ruin* t *with initial reserves* u *and deficit at ruin less than* y *is:*

$$\psi^{**}_{m,n,y} = \int_0^\infty \int_0^\infty e^{-mt-nu}\ \psi_{t,u,y}\ du\ dt = \frac{a^*(m-cn)\, B^*_y(n)\ - \mathbf{U}(m)\,\mathbf{C}_m(n)}{1 - b^*(n)\,a^*(m-cn)} \tag{2.1}$$

where

$$B^*_y(n) = \int_{a=0}^\infty \left(B(a+y)\ - B(a)\right)\ e^{-na}\, d\,a \tag{2.2}$$

the ***structure column*** *vector is given by*

$$\mathbf{C}_m(n) = \boldsymbol{a}^*[m-cn] = ((m-cn)\mathbf{I} - \mathbf{A})^{-1}\ \mathbf{a}$$

and the row vector $\mathbf{U}(m)$ *is defined within the proof of the theorem in (**??**).*

b) The vector $\mathbf{U}(m)$ *may be obtained as the solution of a system whose structure depends on the multiplicity of the roots of the Cramer-Lundberg equation:*

$$1 - b^*(n)\,a^*(m-cn) = 0 \tag{2.3}$$

Let q denote the number of non-negative distinct roots $\{n_j(m)\}_{j=1}^{q}$, of (??), and let $\sigma_{mj}, j = 1, ..., q$ denote their respective multiplicities. Let $f^{k)}(n)$ denote the k'th derivative with respect to n of a function $f(n)$, let $D_{m,y}(n) = B_y^*(n)\, a^*(m - cn)$ and let $\mathbf{D}_{m,y}$ be the row vector

$$\mathbf{D}_{m,y} = \left(D_{m,y}^{0)}(n_{m1}), \ldots, D_{m,y}^{\sigma_{m1}-1)}(n_{m1}), \ldots, D_{m,y}^{0)}(n_{mq}), \ldots, D_{m,y}^{\sigma_{mq}-1)}(n_{mq})\right)$$

Then, the vector $\mathbf{U}(m)$ is given by:

$$\mathbf{U}(m) = \mathbf{D}_{m,y}\, \mathbf{C}_m^{-1}$$

where the ***structure matrix*** $\mathbf{C}_m$ *is given columnwise by:*

$$\mathbf{C}_m = \left(\mathbf{C}_m^{0)}(n_{m1}), \ldots, \mathbf{C}_m^{\sigma_{m1}-1)}(n_{m1}), \ldots, \mathbf{C}_m^{0)}(n_{mq}), \ldots, \mathbf{C}_m^{\sigma_{mq}-1)}(n_{mq})\right)$$

Therefore, we may rewrite 2.1 in the alternative fashion

$$\psi_{m,n,y}^{**} = \frac{B_y^*(n)\, a^*(m - cn) - \mathbf{D}_{m,y}\, \mathbf{C}_m^{-1}\, \mathbf{C}_m(n)}{1 - b^*(n)\, a^*(m - cn)}$$

c) The single Laplace transform of the density function of the time to ruin t **with initial reserves** 0 *and deficit at ruin less than y (which coincides with the c.d.f. of the downladder height distribution) is:*

$$G_m^{(-)}(y) = \Psi'^{*}_{m,0,y} = \left(\frac{1}{c}\right) \mathbf{D}_{m,y}\, \mathbf{C}_m^{-1}\, \mathbf{a} \tag{2.4}$$

Note: The formula for the ladder height distribution $G_m^{(-)}(y)$ will be further explicitized in the case the claims distribution is phase–type as well.
Proof. a) Applying the double-Laplace transform operator

$$\psi_{m,n,y}^{**} = \int_0^\infty \int_0^\infty e^{-mt-nu}\ \psi_{t,u,y}\, du\, dt$$

to 1.2 and using lemmas 3 and 4 in the Appendix we find that that

$$\begin{aligned} \psi_{m,n,y}^{**} = &\ B_y^*(n)\, a^*(m - cn) - \mathbf{H}(m)\, \mathbf{a}^*[m - cn] \\ &+ \psi_{m,n,y}^{**}\, b^*(n)\, a^*(m - cn)\ - \mathbf{G}(m)\ \mathbf{a}^*[m - cn] \\ = &\ \psi_{m,n,y}^{**}\, b^*(n)\, a^*(m - cn)\ + B_y^*(n)\, a^*(m - cn) - \mathbf{U}(m)\, \mathbf{C}_m(n) \end{aligned}$$

where

$$\mathbf{U}(m) = \mathbf{G}(m) + \mathbf{H}(m) \tag{2.5}$$

and $\mathbf{C}_m(n) = \boldsymbol{a}^*[m - cn]$ by definition.

b) To obtain the alternative formula for $\mathbf{U}(m)$ we use the fact that as the Laplace transform of a positive random variable, the function $\psi^{**}_{m,n,y}$ is finite for any value of $n, \mathrm{Re}(n) \geq 0$, and hence that at $n_{mj}, j = 1, ..., q$ the numerator of 2.1 must have roots of the same multiplicity as those of the denominator. This yields the following system of r equations and unknowns

$$\mathbf{U}(m)\,\mathbf{C}^{l)}_m(n_{mj}) \;=D^{l)}_{m,\,y}(n_{mj})\,; \quad j = 1, ..., q \quad l = 0,\, ..., \sigma_{mj} - 1$$

where $\mathbf{C}^{l)}_m(n_{mj})$ and $D^{l)}_{m,\,y}(n_{mj})$ are the l-th derivatives with respect n at n_{mj}, or, using matrix notation

$$\mathbf{U}(m)\,\mathbf{C}_m \;=\; \mathbf{D}_{m,y} \;\Rightarrow\; \mathbf{U}(m) = \mathbf{D}_{m,y}\,\mathbf{C}_m^{-1}$$

c) We obtain the c.d.f. of the ladder height using the Tauberian theorem

$$\begin{aligned}
\psi^{*}_{m,0,y} &= \lim_{n\to\infty} n\,\psi^{**}_{m,n,y} \\
&= \lim_{n\to\infty} n\,\frac{B^*_y(n)\,a^*(m-cn) - \mathbf{D}_{m,y}\,\mathbf{C}_m^{-1}\,\mathbf{C}_m(n)}{1 - b^*(n)a^*(m-cn)} \\
&= -\,\mathbf{D}_{m,y}\,\mathbf{C}_m^{-1}\lim_{n\to\infty} n\;\mathbf{C}_m(n) = \left(\frac{1}{c}\right)\mathbf{D}_{m,u}\,\mathbf{C}_m^{-1}\,\mathbf{a}
\end{aligned}$$

because

$$\begin{aligned}
\lim_{n\to\infty} n\,\mathbf{C}_m(n) &= \lim_{n\to\infty} n\,\mathbf{a}^*\,[m-cn] \\
&= \lim_{n\to\infty} n\;\;\left((m-cn)\mathbf{I} - \mathbf{A}\right)^{-1}\,\mathbf{a} = -\left(\frac{1}{c}\right)\mathbf{a}
\end{aligned}$$

■

Note: This result reduces the ruin problem to the computation of the roots with nonnegative part of the Cramer Lundberg equation and that of $\mathbf{C}_m^{-1}\,\mathbf{C}_m(n)$. In the particular cases discussed in the introduction, the matrices $\mathbf{C}_m^{-1}$ to be inverted turn out to be quite well known.

Example 1: In the *hyperexponential* case, the *structure column* vector is given by

$$\mathbf{C}_m(n) = \mathbf{a}^*[m-cn] = \begin{pmatrix} \frac{\nu_1}{\nu_i+m-cn} \\ \vdots \\ \frac{\nu_r}{\nu_r+m-cn} \end{pmatrix}$$

and when the non-negative roots of the Lundberg's equation are distinct, we will face a Cauchy $r \times r$ *structure matrix*

$$\mathbf{C}_m = diag\,\{\nu_i\}\left\{(m+\nu_i-cn_{mj})^{-1}\right\}_{i,j}$$

with explicit inverse, see Schechter(1959)

$$\mathbf{C}_m^{-1} = \left\{ \frac{\prod_{k=1}^{l} (m + \nu_j - cn_{mk})}{\prod_{k=1(k\neq i)}^{l} (\nu_j - \nu_k)} \cdot \frac{1}{(m + \nu_j - cn_{mi})} \cdot \frac{\prod_{k=1}^{l} (m + \nu_k - cn_{mi})}{\prod_{k=1(k\neq i)}^{l} c\,(n_{mk} - n_{mi})} \right\}_{i,j} \times diag \left\{ \frac{1}{\nu_i} \right\}$$

Example 2: For the single Erlang(p), the *structure column* vector can be written

$$\mathbf{C}_m(n) = \mathbf{a}^*[m - cn] = \begin{pmatrix} \left(\frac{\nu}{\nu+m-cn}\right)^{p+1-1} \\ \vdots \\ \left(\frac{\nu}{\nu+m-cn}\right)^{p+1-p} \end{pmatrix}$$

and we find a Vandermonde $p \times p$ *structure matrix*

$$\mathbf{C}_m = diag\,\{\nu\} \left\{ (m + \nu - cn_{mj})^{-i} \right\}_{i,j}$$

and, for instance, in the particular case of Erlang(2) and non-negative distinct roots of the Lundberg's equation

$$\mathbf{C}_m = diag\,\{\nu\} \begin{pmatrix} (m + \nu - cn_{m1})^{-1} & (m + \nu - cn_{m2})^{-1} \\ (m + \nu - cn_{m1})^{-2} & (m + \nu - cn_{m2})^{-2} \end{pmatrix}$$

with explicit inverse

$$\mathbf{C}_m^{-1} = \left(\frac{1}{cn_{m1} - cn_{m2}} \right) \begin{pmatrix} (m + \nu - cn_{m1})^2 & -(m + \nu - cn_{m2})(m + \nu - cn_{m1})^2 \\ -(m + \nu - cn_{m2})^2 & (m + \nu - cn_{m2})^2 (m + \nu - cn_{m1}) \end{pmatrix} \times diag \left\{ \frac{1}{\nu} \right\}$$

And also for a multiple root

$$\mathbf{C}_m = diag\,\{\nu\} \begin{pmatrix} (m + \nu - cn_{m1})^{-1} & c\,(m + \nu - cn_{m1})^{-2} \\ (m + \nu - cn_{m1})^{-2} & 2\,c\,(m + \nu - cn_{m1})^{-3} \end{pmatrix}$$

$$\mathbf{C}_m^{-1} = \begin{pmatrix} 2\,(m + \nu - cn_{m1}) & -(m + \nu - cn_{m1})^2 \\ -(m + \nu - cn_{m1})^2 / c & (m + \nu - cn_{m1})^3 / c \end{pmatrix} diag \left\{ \frac{1}{\nu} \right\}$$

Example 3: For the *Hypercoxian* distributions, the inverse $\mathbf{C}_m^{-1}$ has a block structure with components given in the previous example.

In the case the claim sizes are of phase-type as well, a closed formula for the **single Laplace transform** of the multivariate ruin probability may be inferred from Theorem 4.1. by Asmussen(1992) (see (2.7) below). This formula exploits the fact that the ruin probability is also phase–type, and is determined completely by a vector $\boldsymbol{\eta}_m$ giving the probabilities of downcrossing in the various phases at the first downladder time. Asmussen represented the vector $\boldsymbol{\eta}_m$ in terms of the roots with negative real part of the Cramer-Lundberg's equation $1 - b^*(n)\ a^*(m - cn) = 0$; in the case of distinct negative roots $\gamma_i, i = 1, ...M$ his result is equivalent to:

$$\boldsymbol{\eta}_m = \mathbf{1}\mathbf{S}^{-1} \tag{2.6}$$

where $\mathbf{S}$ is a matrix with columns $\boldsymbol{b}^*[\gamma_i], i = 1, ...M$ (see Asmussen(1992) and Avram and Usábel(2002)). We provide here an alternative representation in terms of the roots with non-negative real part (see (2.8)), considering also the case of multiple roots.

Corollary 2. *a) Under the assumption of PH($\boldsymbol{\beta}, \mathbf{B}$) claim sizes, the single Laplace transform in time (or "exponentially killed ruin probability") with initial reserves u and deficit at ruin less than y is:*

$$\psi^*_{m,u,y} = \boldsymbol{\eta}_m\, e^{\mathbf{Q}_m\, u} \left(\mathbf{I} - e^{\mathbf{B}y}\right)\, \mathbf{1} \tag{2.7}$$

where

$$\mathbf{Q}_m = \mathbf{B} + \mathbf{b}\boldsymbol{\eta}_m; \qquad \mathbf{b} = -\mathbf{B}\,\mathbf{1}$$

b) If in addition the waiting times are phase–type as well (1.5), then

$$\boldsymbol{\eta}_m = \left(\frac{1}{c}\right)\, \boldsymbol{\Delta}_m\, \mathbf{C}_m^{-1}\, \mathbf{a} \tag{2.8}$$

where

$$\boldsymbol{\Delta}_m = \left(\Delta_m^{0)}(n_{m1}), \ldots, \Delta_m^{\sigma_{m1}-1)}(n_{m1}), \ldots, \Delta_m^{0)}(n_{mq}), \ldots, \Delta_m^{\sigma_{mq}-1)}(n_{mq})\right)$$

and

$$\Delta_m^{0)}(n) = a^*(m - cn)\ \ \boldsymbol{\beta}\,(n\mathbf{I} - \mathbf{B})^{-1}$$

Note The alternative formula (2.8) for $\boldsymbol{\eta}$ will be more easily implemented in case the order of the claim size representation is higher then that of the interarrivals, like for example in the classical Levy case.

Proof. Part a) may be inferred from Asmussen(1992), where the different representation (2.6) for $\boldsymbol{\eta}$ was obtained, assuming distinct roots of the Cramer-Lundberg equation (see also and Avram and Usábel(2002)).

To obtain part b), it is sufficient to establish the particular case $u = 0$ which yields the ladder height distribution (see for example Asmussen(2000) chapter 8). It remains now only to apply Theorem 1c), taking into account that under PH($\boldsymbol{\beta}, \mathbf{B}$) claim sizes,

$$B_y^*(n) = \boldsymbol{\beta}\,(n\mathbf{I} - \mathbf{B})^{-1}\,(\mathbf{I} - e^{\mathbf{B}y})\mathbf{1}$$

and therefore

$$D_{m,y}(n) = \Delta_m^{0)}(n)\ (\mathbf{I} - e^{\mathbf{B}y})\mathbf{1}$$

Since the last two terms in the RHS above are independent of n, they may be factored outside the vector $\mathbf{D}_{m,y}$ (which is constructed out of $D_{m,y}(n)$ by differentiating and substituting n), yielding:

$$\psi^*_{m,0,y} = \boldsymbol{\eta}_m\ \left(\mathbf{I} - e^{\mathbf{B}y}\right)\ \mathbf{1}$$

with $\boldsymbol{\eta}_m$ given by (2.8). ■

References

[1] Asmussen, S.(1984) *Approximations for the probability of ruin within finite time.* Scandinavian Actuarial Journal, 31-57

[2] Asmussen, S.(1987) *Applied Probability and Queues.* Wiley, New York.

[3] Asmussen and Rolski(1991) *Computational Methods in risk theory: a matrix-algorithmic approach.* Insurance:Mathematics and Economics, Vol. 10, 259-274

[4] Asmussen, S.(1992) *Phase-type representations in random walk and queueing problems.* The Annals of Probability, Vol. 20, No. 2, 772-789

[5] Asmussen, S. and Hojgaard, B. (1999) *Approximations for finite time ruin probabilities in the renewal model.* Scandinavian Actuarial Journal, **99**, 106-119.

[6] Avram, F. and Usabel, M. (2001) *Finite time ruin probabilities with one Laplace inversion.* Submitted to *Insurance: Mathematics and Economics*

[7] S. Asmussen, F. Avram, M. Usabel (2001). Erlangian approximations for finite–horizon ruin probabilities. Submitted to Astin Bulletin

[8] Avram, F. and Usabel, M. (2002) Finite time ruin probabilities via the semi-Markov embedding. Preprint.

[9] Bohman, H. (1971) Ruin probabilities. Skandinavisk Aktuarietidskrift. 159-163.

[10] Bohman, H. (1974) Fourier Inversion-Distribution functions-Long tails. Scand. Actuarial Journal. 43-45

[11] Bohman, H. (1975) Numerical inversion of characteristic functions. Scan. Act. Journal. 121-124

[12] Cramèr, H. (1955) Collective Risk Theory. Jubille Volume of F. Skandia.

[13] Dickson, D. and Egidio dos Reis, A. (1996). On the distribution of the duration of negative surplus. Scandinavian Actuarial Journal, 148-164

[14] Egidio dos Reis, A. (1993). How long is the surplus below zero? Insurance: Mathematics and Economics, 12, 23-38.

[15] Egidio dos Reis, A. (2000). On the moments of ruin and recovery times. Insurance: Mathematics and Economics, 27, 331-343.

[16] Gerber, H. , Goovaerts, M. & Kaas, R. (1987) On the probability and severity of ruin. ASTIN Bulletin 17, 151-163.

[17] Neuts, M. (1975) Probability distibutions of phase-type. Liber Amicorum Professor Emeritus H. Florin, Dep. of Mathematics, University of Louvain, Belgium, 173-206.

[18] Neuts, M. (1977) Algorithms for the waiting time distribution under various queue disciplines in the M / G / 1 queue with service time distribution of phase-type. TIMS Studies in the Management Science, Vol. 7, 177-197

[19] Piessens, R. (1969) New quadrature formulas for the numerical inversion of Laplace transforms. BIT 9, 351-361

[20] Seal, H. (1971) Numerical calculation of the Bohman-Esscher family of convolution-mixed negative binomial distribution functions. Mitt. Verein. schweiz. Versich.-Mathr. 71, 71-94.

[21] Seal, H. (1974) The numerical calculation of U(w,t), the probability of non-ruin in an interval (0,t). Scan. Act. Journal, 121-139.

[22] Seal, H. (1977) Numerical inversion of characteristic functions. Scan. Act. Journal, 48-53.

[23] Stanford, D. and Stroinski, K. (1994) Recursive methods for computing finite-time ruin probabilities for phase-type distributed claim sizes. Astin Bulletin 24(2), 235-254.

[24] Stanford, D., Stroinski, K. and Lee, K. (2000) Ruin probabilities based at claim instants for some non-Poisson claim processes. Insurance: Mathematics and Economics, 26, 251-267.

[25] Thorin, O. (1970). Some remarks on the ruin problem in case the epochs of claims form a renewal process. Skandinavisk Aktuarietidskrift. 29-50.

[26] Thorin, O. (1971). Further remarks on the ruin problem in case the epochs of claims form a renewal process. Skandinavisk Aktuarietidskrift. 14-38, 121-142.

[27] Thorin, O. (1973). The ruin problem in case the tail of a distribution is completely monotone. Skandinavisk Aktuarietidskrift. 100-119.

[28] Thorin, O. (1977). Ruin probabilities prepared for numerical calculation. Scandinavian Actuarial Journal.

[29] Thorin, O. (1982). Probabilities of Ruin. Scandinavian Actuarial Journal. 65-102.

[30] Thorin, O. and Wikstad, N. (1973). Numerical evaluation of ruin probabilities for a finite period. ASTIN Bulletin VII:2, 138-153.

[31] Wikstad,N. (1971) Exemplifications of ruin probabilities. ASTIN Bulletin, vol. VI, part 2.

[32] Wikstad, N. (1977). How to calculate Ruin probabilities according to the classical Risk Theory. Scand. Actuarial Journal.

[33] Willmot, G. (1998) On a class of approximations for ruin and waiting time probabilities. Oper. Res. Lett. 22, 27-32

[34] Willmot, G. (1999) A Laplace transform representation in a class of renewal queueing and risk process. J. Appl. Prob. 36, 570-584.

[35] Willmot, G. (2000) On the evaluation of the conditional distribution of the deficit at the ruin time. Scandinavian Actuarial Journal, 100(1) 63-79.

Lemma 3. *The double Laplace transform of the double integral*

$$\begin{aligned}&\int_{s=0}^{t}\int_{z=0}^{u+cs} F(t-s,u+cs-z)\,b(z)\,a(s)\,dzds\\ &= F^{**}(m,n)\,b^{*}(n)\;\;a^{*}(m-cn)-\boldsymbol{\alpha}\;\mathbf{G}(m)\;\mathbf{a}^{*}\left[m-cn\right]\end{aligned}$$

where $a(t) = \boldsymbol{\alpha}\, e^{\mathbf{A}\,t}\,\mathbf{a}$ *with Laplace transform* $a^{*}(n) = \boldsymbol{\alpha}\;\mathbf{a}^{*}\left[n\right]$ *is a phase–type density where*

$$\begin{aligned}\mathbf{a}^{*}\left[m-cn\right] &= \left((m-cn)\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{a}\\ \mathbf{a} &= -\mathbf{A}\,\mathbf{1}\end{aligned}$$

and

$$\mathbf{G}(m) = \ \boldsymbol{\alpha}\, F^{**}\left(m, \frac{m\mathbf{I}-\mathbf{A}}{c}\right)\, b^{*}\left(\frac{m\mathbf{I}-\mathbf{A}}{c}\right)$$

Proof.

Using the definition of the double-Laplace transform

$$\int_{t=0}^{\infty}\int_{u=0}^{\infty}\left(\int_{s=0}^{t}\int_{z=0}^{u+cs} F(t-s, u+cs-z)\, b(z)\, a(s)\, dzds\right) e^{-mt-nu}\ dudt$$

we find after the change of variables $t-s=w, \quad u+cs-z=v, \quad s=s, \quad z=z$ that it equals:

$$\begin{aligned}
&= \int_{s=0}^{\infty}\int_{z=0}^{\infty}\int_{w=0}^{\infty}\int_{v=(cs-z)^+}^{\infty} F(w,v)\, b(z)a(s)e^{-(m-cn)s-nz-mw-nv}\ dvdwdzds \\
&= \int_{s=0}^{\infty}\int_{z=cs}^{\infty}\int_{w=0}^{\infty}\int_{v=0}^{\infty} F(w,v)\, b(z)a(s)e^{-(m-cn)s-nz-mw-nv}\, dvdwdzds \\
&+ \int_{s=0}^{\infty}\int_{z=0}^{cs}\int_{w=0}^{\infty}\int_{v=cs-z}^{\infty} F(w,v)\, b(z)a(s)e^{-(m-cn)s-nz-ma-nb}\, dvdwdzds \\
&= \int_{w=0}^{\infty}\int_{v=0}^{\infty} F(w,v)\ e^{-mw-nv}\int_{z=0}^{\infty} b(z)e^{-nz}\int_{s=0}^{z/c} a(s)e^{-(m-cn)s}\ dsdzdvdw \\
&+ \int_{w=0}^{\infty}\int_{v=0}^{\infty} F(w,v)\ e^{-mw-nv}\int_{z=0}^{\infty} b(z)e^{-nz}\int_{s=z/c}^{(v+z)/c} a(s)e^{-(m-cn)s}\ dsdzdvdw \\
&= \int_{w=0}^{\infty}\int_{v=0}^{\infty} F(w,v)\ e^{-mw-nv}\int_{z=0}^{\infty} b(z)e^{-nz}\int_{s=0}^{(v+z)/c} a(s)e^{-(m-cn)s}\ dsdzdvdw
\end{aligned} \tag{.9}$$

Under the phase–type assumption, the inner integral is:

$$\begin{aligned}
&\int_{s=0}^{(v+z)/c} a(s)e^{-(m-cn)s}\ ds = \boldsymbol{\alpha}\int_{s=0}^{(v+z)/c} e^{s\mathbf{A}}e^{-(m-cn)s}\ ds\ \mathbf{a} \\
&= \boldsymbol{\alpha}\left(\mathbf{I} - e^{-\frac{v+z}{c}((m-cn)\mathbf{I}-\mathbf{A})}\right)((m-cn)\mathbf{I}-\mathbf{A})^{-1}\ \mathbf{a}
\end{aligned}$$

substituting into .9 we obtain the result.

$$F^{**}(m,n)\, b^{*}(n)\ \ a^{*}(m-cn) - \mathbf{G}(m)\, \mathbf{a}^{*}\left[m-cn\right]$$

where the row vector $\mathbf{G}(m)$ is given by

$$\begin{aligned}
\mathbf{G}(m) &= \ \boldsymbol{\alpha}\int_{w=0}^{\infty}\int_{v=0}^{\infty} F(w,v)\ e^{-mw}\, e^{-(m\mathbf{I}-\mathbf{A})\frac{v}{c}}\int_{z=0}^{\infty} b(z)\, e^{-(m\mathbf{I}-\mathbf{A})\frac{z}{c}}\ dzdvdw \\
&= \boldsymbol{\alpha}\, F^{**}\left(m, \frac{m\mathbf{I}-\mathbf{A}}{c}\right)\ b^{*}\left(\frac{m\mathbf{I}-\mathbf{A}}{c}\right)
\end{aligned}$$

■

Lemma 4. *The double Laplace transform of the function*

$$(B(u+ct+y)-B(u+ct))\ \ a(t)$$

where $a(t) = \ \boldsymbol{\alpha}\, e^{\mathbf{A}\,t}\,\mathbf{a}$ *is a phase–type density can be given by*

$$a^*(m-cn)\, B_y^*(n) - \mathbf{H}(m)\, \mathbf{a}^*\,[m-cn]$$

where

$$\begin{aligned} B_y^*(n) &= \int_{a=0}^{\infty} (B(a+y)\ -B(a))\ \ e^{-na}\ da \\ \mathbf{H}(m) &= \ \boldsymbol{\alpha} \int_{a=0}^{\infty} (B(a+y)\ -B(a))\ \ \ e^{-(m\mathbf{I}-\mathbf{A})\frac{a}{c}} da \\ &= \ \boldsymbol{\alpha}\, B_y^* \left(\frac{m\mathbf{I}-\mathbf{A}}{c}\right) \end{aligned}$$

and

$$\begin{aligned} \mathbf{a}^*\,[m-cn] &= ((m-cn)\mathbf{I}-\mathbf{A})^{-1}\ \mathbf{a} \\ \mathbf{a} &= -\mathbf{A}\,\mathbf{1} \end{aligned}$$

Proof. For $a(t) = \boldsymbol{\alpha}\, e^{\mathbf{A}t}\,\mathbf{a}$ we can obtain the double-Laplace transform

$$\int_0^{\infty}\int_0^{\infty} (B(u+ct+y)\ -B(u+ct))\ \boldsymbol{\alpha}\, e^{\mathbf{A}\,t}\,\mathbf{a}\ e^{-mt-nu} du\, dt$$

After the change of variables

$$u+ct = a \quad\ t = t$$

and shifting the order of integration we easily get

$$\int_{a=0}^{\infty} (B(a+y)\ -B(a))\ e^{-na} \int_{t=0}^{(a/c)} \boldsymbol{\alpha}\, e^{-((m-cn)\mathbf{I}-\mathbf{A})t}\mathbf{a}\ \ dt\, da$$

$$= \boldsymbol{\alpha} \int_{a=0}^{\infty} (B(a+y)\ -B(a))\ \ e^{-na}\ \left(\mathbf{I} - e^{-((m-cn)\mathbf{I}-\mathbf{A})\frac{a}{c}}\right)\ da\ \ ((m-cn)\,\mathbf{I}-\mathbf{A})^{-1}\,\mathbf{a}$$

and obtain the result

$$a^*(m-cn)\, B_y^*(n) - \mathbf{H}(m)\, \mathbf{a}^*\,[m-cn]$$

■

Miguel Usábel
Dept. of Business Administration
Universidad Carlos III de Madrid
Avda. Universidad Carlos III, 22
28270 Colmenarejo(Madrid)
Spain
E-mail: usabel@emp.uc3m.es