The two barriers ruin problem via a Wiener Hopf decomposition approach

Florin Avram* Martijn R. Pistorius† Miguel Usabel‡

ABSTRACT

Consider an insurance company whose capital U evolves as a risk processes with phase–type inter-arrivals and claims. In this note we study the probability and severity of ruin before the capital U reaches an upper barrier $K > 0$. The main tools we use are Asmussen and Kella's embedding [5, 6] and Wiener-Hopf factorization of generator matrices.

Keywords: Two sided exit problem, phase–type distributions, semi-Markov embedding, Wiener-Hopf factorization of matrices, renewal process.

AMS Subject Classification: 60G40, 90A09.

1 Introduction

Consider an insurance company, whose capital is modeled by a positive drift added to a pure jump process with negative jumps. The drift, say p, models the premium income stream and the jumps stand for the claims the company receives. One is interested in the time and severity of ruin. The transform analytic approach to this problems, going back to Cramèr [12] and Sparre Andersen [1], consists in formulating integro-differential equations or renewal equations for the functions of interest and solving them via a double Laplace - Stieltjes transform (to be inverted numerically).

For example, if the jump process is a compound Poisson process with arrival rate λ and density of the jump-distribution $b(z)$, the single Laplace transform in time $\Psi(u) = \mathbb{E}_u[\mathrm{e}^{-\delta\tau}]$ of the time of ruin τ satisfies the integro-differential equation

$$p\Psi'(u) + \lambda\left(\int_0^\infty \Psi(u-z)b(z)\mathrm{d}z - \Psi(u)\right) - \delta\Psi(u) = 0 \qquad u > 0.$$

*Dept. de Math., Université de Pau and Dept. of Actuarial Maths. & Statistics, Heriot-Watt University, Edinburgh EH14 4AS U.K., E-mail: Florin.Avram@univ-Pau.fr

†Corresponding author. Dept. de Math., Utrecht University, P.O. Box 80.010, 3508TA Utrecht, E-mail: pistorius@math.uu.nl

‡Dept. Economia Empresa, Universidad Carlos III de Madrid, Spain, E-mail: usabel@emp.uc3m.es

In general, solving integro-differential equations would require a second Laplace transform in u (compounding therefore the numerical errors of inverting). At the minor expense however of assuming phase–type distributions, it was noticed by Asmussen [5] that it is possible to embed the non-Markovian renewal model into a piecewise deterministic Markov modulated model with a finite order Markovian environment (also called fluid model), which, as explained below, replaces the integro-differential equation by a first order differential system. In the case the parameters are constant, this system is solvable explicitly via matrix spectral or Wiener-Hopf decompositions, removing the need for the second Laplace transform in u. In this note we illustrate this fluid embedding/ODE approach by stating (Section 2) and solving (Section 3) the two barrier ruin problem with phase type distributions for the interarrival times and the jumps. To our knowledge the obtained result is new in the literature (see though Proposition 1.6, Chapter XI in [5] for the more restrictive perpetual skipfree case).

2 Model and problem

Before we introduce the model for the reserves process U, we first review some properties of phase type distrubtions.

2.1 Phase-type distributions

A distribution F on $(0, \infty)$ is *phase-type* if it is the distribution of the absorbtion time ζ in a finite continuous time Markov process $J = \{J_t\}_{t \geq 0}$ with one state Δ absorbing and the remaining ones $1, \dots, m$ transient. That is, $F(t) = \mathbb{P}(\zeta \leq t)$ where $\zeta = \inf\{s > 0 : J_s = \Delta\}$. The parameters are m, the restriction of $\boldsymbol{T}$ of the full intensity matrix to the m transient states and the initial probability (row) vector $\boldsymbol{\alpha} = (\alpha_1 \dots \alpha_m)$ where $\alpha_i = \mathbb{P}(J_0 = i)$. For any $i = 1, \dots, m$, let t_i be the intensity of a transition $i \to \Delta$ and write $\boldsymbol{t} = (t_1 \dots t_m)'$ for the (column) vector of such intensities. Further, let $\mathbf{1}_m$ denote a $m \times 1$ column vector of ones. It follows that the cumulative distribution function F is given by $F(x) = 1 - \boldsymbol{\alpha} \mathrm{e}^{\boldsymbol{T}x} \mathbf{1}_m$.

Phase-type distributions include and generalize exponential distributions in series and/or parallel and form a dense class in the set of all distributions on $(0, \infty)$. Much of the applicability of the class comes from the probabilistic interpretation, in particular the fact that that the overshoot distributions $F(x + y)/(1 - F(x))$ belong to a finite vector space (in fact, the overshoot distribution is again phase–type with the same m and $\boldsymbol{T}$ but α_i replaced by $\mathbb{P}(J_x = i | \zeta > x)$). This provides matrix analogues of many formulas for the exponential distribution ($m = 1$) based upon the memoryless property. See for example [4, 5] for surveys on phase type distributions.

2.2 Renewal risk model

We model the reserves process U of an insurance company by a process of the form

$$U_t = u - \sum_{k=1}^{N_t} Z_k + p\,t \tag{1}$$

where $p > 0$ is the premium rate, the claims $Z_1, Z_2, \ldots$ are i.i.d. random variables with common distribution $B(z)$ and N_t is an independent renewal process with inter-arrival distribution $A(z)$ and mean inter-arrival time denoted by λ^{-1}, and the distributions $A(z), B(z)$ are concentrated on $(0, \infty)$ and of phase-type $(m, \boldsymbol{\alpha}, \boldsymbol{A})$, and $(n, \boldsymbol{\beta}, \boldsymbol{B})$, respectively.

From now on we will assume that $p = 1$. Results for the original case with $p \neq 1$ can then be obtained by replacing $(u, \boldsymbol{A})$ by $p^{-1}(u, \boldsymbol{A})$.

For the process U we will solve the ruin problem with an upper barrier K. Let

$$\tau(K) = \inf\{t > 0 : U_t \notin [0, K)\}$$

be the first time the reserves process U reaches the level K or downcrosses zero. The problem we will solve is to find expressions for the time Laplace transform of the probability that U reaches K before time t and before it has droped below zero

$$\Psi_\delta^+(u, K) = \mathbb{E}_u[\mathrm{e}^{-\delta\tau(K)}; U_{\tau(K)} = K] \tag{2}$$

and viceversa

$$\Psi_\delta^-(u, K) = \mathbb{E}_u[\mathrm{e}^{-\delta\tau(K)}; U_{\tau(K)} \leq 0],$$

where we write $\mathbb{E}_u$ for the expectation conditioned on $\{U_0 = u\}$ and $\mathbb{E}_u[\mathrm{e}^{-\delta\tau}; A]$ to denote $\mathbb{E}_u[\mathrm{e}^{-\delta\tau}\mathbf{1}_A]$ with $\mathbf{1}_A$ the indicator of the event A. Below we will solve well as the time Laplace transform of the joint probabilities of ruin with upper-barrier before t and severity of ruin larger than y:

$$\Psi_\delta^-(u, K, y) = \mathbb{E}_u[\mathrm{e}^{-\delta\tau(K)}; U_{\tau(K)} \leq -y]. \tag{3}$$

In the next section we will find explicit expressions for these quantities.

3 Solution

3.1 The semi-Markov embedding

We now replace the original model with jumps U by an equivalent continuous Markov process (U', J). Denote by E^+ and E^- the phases of the interarrival distribution A and of the downward jumps B. Informally, we can get the process (J, U') from U by levelling out the negative jumps to sample path segments with slope -1, and setting J equal to the current phase of the underlying Markov process of the interarrival time or jump (see Section 2.1).

To be more precise, in the Markov embedding (J, U'), J is a finite state Markov process with state space $E = E^+ \cup E^-$ and generator

$$\boldsymbol{G} = \left(\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{a} \otimes \boldsymbol{\beta} \\ \hline \boldsymbol{b} \otimes \boldsymbol{\alpha} & \boldsymbol{B} \end{array} \right)$$

Let $p : E \to \{-1, 1\}$ take the value ± 1 on $E^\pm$. Then the additive functional U' is defined by

$$U'_t = u + \int_0^t p(J_s)\mathrm{d}s.$$

In the literature (e.g. [14]) this model for (J, U') is called a fluid model. Setting $\boldsymbol{P} \equiv \mathrm{diag}(p(i))$, we observe that the generator of (J, U') is given by

$$\boldsymbol{G}\boldsymbol{f} + \boldsymbol{P}\boldsymbol{f}'$$

for functions $\boldsymbol{f} \in C^1(\mathbb{R}^E)$. Note that in the vector $\boldsymbol{f}(u)$, the coordinate indicates the state of J at time u.

3.2 The ODE system for the embedded process

Denote by τ' the first time that U' exits $[0, K)$. By sample path comparison we see that $\tau(K) = T(\tau')$ where

$$T(t) = \int_0^t I(J_s \in E^+)\mathrm{d}s$$

indicates the time up to time t that J is in an upcrossing phase. Let $\boldsymbol{M}(u)$ denote the $|E| \times |E|$ matrix, which has as ijth element $(i, j \in E)$ $\boldsymbol{M}(u)_{ij} = \mathbb{E}_{i,u}[\mathrm{e}^{-\delta T(\tau')}; J_{\tau'} = j]$, where $\mathbb{E}_{i,u}$ denotes expectation conditioned on $U'_0 = u$ and $J_0 = i \in E$. Without loss of generality we assume that the first $|E^+|$ rows and columns of $\boldsymbol{M}(u)$ correspond to states in E^+; we write $\boldsymbol{M}_1(u)$ $(\boldsymbol{M}_2(u))$ for the first $|E^+|$ (last $|E^-|$) rows of $\boldsymbol{M}(u)$. Note that finding the matrix $\boldsymbol{M}_1(u)$ yields the probabilities (2)–(3). Indeed, breaking $\boldsymbol{M}_1(u) = (\boldsymbol{\Psi}^+(u), \boldsymbol{\Psi}^-(u))$ into the states corresponding to E^+ and E^- respectively (corresponding to "upper barrier first" and "ruin first" probabilities, respectively), we have $\Psi_\delta^+(u, K) = \boldsymbol{\alpha}\boldsymbol{\Psi}^+(u)\mathbf{1}_m$ and $\Psi_\delta^-(u, K) = \boldsymbol{\alpha}\boldsymbol{\Psi}^-(u)\mathbf{1}_n$.

Let $\boldsymbol{I}_\delta$ denote the $|E| \times |E|$ diagonal matrix with the ith element on the diagonal given by δ if $i \in E^+$ and 0 otherwise and write δ_{ij} for the Kronecker delta (which is 1 if $i = j$, 0 else).

Lemma 1. *The matrix $\boldsymbol{M}(u)$ is the unique solution of the Feynman-Kac equation:*

$$\boldsymbol{P}\boldsymbol{M}'(u) + \boldsymbol{G}\boldsymbol{M}(u) - \boldsymbol{I}_\delta\boldsymbol{M}(u) = \boldsymbol{O} \qquad u \in (0, K) \tag{4}$$

$$\boldsymbol{M}(K)_{ij} = \delta_{ij} \text{ for } i \in E^+, j \in E \qquad \boldsymbol{M}(0)_{ij} = \delta_{ij} \text{ for } i \in E^-, j \in E \tag{5}$$

Proof. Let $u \mapsto \boldsymbol{F}(u)$ be any $|E| \times |E|$ matrix with C^1 components which solves (4) and (5). Then, by Itô's lemma, for each $j \in E$

$$\exp(-\delta T(t))\boldsymbol{F}(U_t)_{J_t,j} \qquad \text{is a martingale}$$

which is bounded on $[0, \tau']$. The optional stopping theorem then implies that

$$\boldsymbol{F}(u)_{ij} = \mathbb{E}_{i,u}[\mathrm{e}^{-\delta T(\tau')}\boldsymbol{F}(U'_{\tau'})_{J'_{\tau},j}].$$

Using the boundary conditions (5) and noting that $J_{\tau'} \in E^+$ if and only if $U'_{\tau'} = K$, we deduce that $\boldsymbol{F}(u) = \boldsymbol{M}(u)$ for $u \in [0, K]$. QED

Now we can rewrite (4) as the first order system

$$\boldsymbol{M}'(u) = \boldsymbol{K}_\delta \boldsymbol{M}(u) \qquad \text{with } \boldsymbol{K}_\delta = -\boldsymbol{P}^{-1}(\boldsymbol{G} - \boldsymbol{I}_\delta) \tag{6}$$

for which we may obtain the matrix exponential solution in terms of the initial value $\boldsymbol{M}(0)$

$$\boldsymbol{M}(u) = \exp(\boldsymbol{K}_\delta\, u)\boldsymbol{M}(0) \tag{7}$$

We could proceed now to work with the matrix exponential solution in terms of the initial values (7) using the spectral decomposition of the matrix $\boldsymbol{K}_\delta$ but an alternative approach more profitable here is to use the Wiener-Hopf factorization of $\boldsymbol{K}_\delta$.

3.3 Wiener-Hopf factorization

In a sequence of papers [11, 13, 16], Barlow, London, Rogers, Williams and McKean investigated Wiener-Hopf factorization of generator matrices. By their results applied to the matrix $\boldsymbol{K}_\delta$ – see for example Theorem 1 and formula (2.7(i)) in Rogers [16]– there exist $|E^\pm| \times |E^\pm|$-generator matrices $\boldsymbol{Q}_\pm$ (nonnegative off-diagonal elements, non-positive row-sums) and $|E^\mp| \times |E^\pm|$-matrices $\boldsymbol{\eta}_\pm$ such that

$$\boldsymbol{K}_\delta \begin{pmatrix} \boldsymbol{I} & \eta_- \\ \boldsymbol{\eta}_+ & \boldsymbol{I} \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & \eta_- \\ \boldsymbol{\eta}_+ & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} -\boldsymbol{Q}_+ & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{Q}_- \end{pmatrix} \tag{8}$$

and if $\boldsymbol{G} - \boldsymbol{I}_\delta$ is transient (which is certainly the case if $\delta > 0$) the matrices $\boldsymbol{Q}_\pm$ are the unique generator matrices satifying (8). **Remarks.** (i) If $\boldsymbol{G} - \boldsymbol{I}_\delta$ is transient, the matrices $\boldsymbol{Q}_\pm$ are the generators of the time-changed Markov processes $\widetilde{J}_\pm$ where $\widetilde{J}_\pm(t) = J(\tau_t^\pm)$ where $\tau_t^\pm = \inf\{s \geq 0 : \pm U'_s > t\}$ is the time-change (note that $\widetilde{J}_\pm$ takes values in $E^\pm$ since U' reaches a new supremum (infimum) in an increasing (decreasing) state).

(ii) The matrices $\boldsymbol{\eta}_\pm$ give the phase probabilities at the first return at the current level. More precisely, they are the initial distribution of $\widetilde{J}_\pm$: $\boldsymbol{\eta}_\pm(i,j)$ is the probability that $\widetilde{J}_\pm(0)$, conditioned on $J_0 = i \in E^\mp$, is in state $j \in E^\pm$ at the first time that $\pm U'$ reaches a new supremum.

(iii) It is clear probabilistically that the generators $\boldsymbol{Q}_{\pm}$ are given by

$$\boldsymbol{Q}_{+} = \boldsymbol{A} - \delta\boldsymbol{I} + (\boldsymbol{a}\otimes\boldsymbol{\beta})\boldsymbol{\eta}_{+} \qquad \boldsymbol{Q}_{-} = \boldsymbol{B} + (\boldsymbol{b}\otimes\boldsymbol{\alpha})\boldsymbol{\eta}_{-} \tag{9}$$

where $\boldsymbol{a} = -\boldsymbol{A}\mathbf{1}, \boldsymbol{b} = -\boldsymbol{B}\mathbf{1}$ (cf. Asmussen [3] or Rogers [16]). This also directly follows from the foregoing. Indeed, write $\boldsymbol{\Phi}^{-}(u)$ for the columns of $\boldsymbol{M}_2(u)$ corresponding to E^{-}. Then we note from (i) and (ii) that, with $K = \infty$, $\boldsymbol{\Psi}^{-}(u) = \boldsymbol{\eta}_{-}\exp(\boldsymbol{Q}_{-}u)$ and $\boldsymbol{\Phi}^{-}(u) = \exp(\boldsymbol{Q}_{-}u)$. Following the line of reasoning in the proof of Lemma 1 one shows that the columns of $(\boldsymbol{\Psi}^{-}, \boldsymbol{\Phi}^{-})'$ satisfy equation (4), which yields the equation for $\boldsymbol{Q}_{-}$ in (9) and the equation

$$\boldsymbol{\eta}_{-}\boldsymbol{Q}_{-} + (\boldsymbol{A} - \delta\boldsymbol{I})\boldsymbol{\eta}_{-} + \boldsymbol{a}\otimes\boldsymbol{\beta} = \boldsymbol{O}.$$

The matrices $(\boldsymbol{\eta}_{+}, \boldsymbol{Q}_{+})$ satisfy a similar equation and the first equation of (9). From (iii) we see then that the matrix $\boldsymbol{\eta}_{-}$ satisfies the Ricatti equation

$$\boldsymbol{\eta}_{-}\boldsymbol{B} + \boldsymbol{\eta}_{-}(\boldsymbol{b}\otimes\boldsymbol{\alpha})\boldsymbol{\eta}_{-} + (\boldsymbol{A} - \delta\boldsymbol{I})\boldsymbol{\eta}_{-} + \boldsymbol{a}\otimes\boldsymbol{\beta} = \boldsymbol{O} \tag{10}$$

with a similar one for $\boldsymbol{\eta}_{+}$ (obtained from (10) by interchanging $\boldsymbol{A} - \delta\boldsymbol{I}$ with with $\boldsymbol{B}$ and $\boldsymbol{b}\otimes\boldsymbol{\alpha}$ with $\boldsymbol{a}\otimes\boldsymbol{\beta}$). In fact, Rogers [16] showed that with $\delta > 0$, $\boldsymbol{\eta}_{-}$ is the unique sub-stochastic solutions of (10).

3.4 Solving the ODE-system

The Wiener-Hopf factorization (8) combined with equation (7) yields then that

$$\begin{pmatrix} \boldsymbol{M}_1(u) \\ \boldsymbol{M}_2(u) \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-\mathbf{Q}_{+}u} & \boldsymbol{\eta}_{-}\mathrm{e}^{\mathbf{Q}_{-}u} \\ \boldsymbol{\eta}_{+}\mathrm{e}^{-\mathbf{Q}_{+}u} & \mathrm{e}^{\mathbf{Q}_{-}u} \end{pmatrix} \begin{pmatrix} \boldsymbol{z}_1 \\ \boldsymbol{z}_2 \end{pmatrix} \tag{11}$$

where $\boldsymbol{z}_1, \boldsymbol{z}_2$ satisfy:

$$\begin{pmatrix} \boldsymbol{I} & \eta_{-} \\ \eta_{+} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_1(0) \\ \boldsymbol{M}_2(0) \end{pmatrix}.$$

Choosing now a new variable $\boldsymbol{z}_1' = \exp(-\boldsymbol{Q}_{+}K)\,\boldsymbol{z}_1$, we find from (5) the following system for the boundary conditions

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{Z}_{-} \\ \boldsymbol{Z}_{+} & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} z_1' \\ z_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{M}_1(K) \\ \boldsymbol{M}_2(0) \end{pmatrix}, \tag{12}$$

where $\boldsymbol{Z}_{\pm} = \boldsymbol{\eta}_{\pm}\exp(\boldsymbol{Q}_{\pm}K)$. Since $\boldsymbol{\eta}_{\pm}$ are substochastic and $\boldsymbol{Q}_{\pm}$ are negative definite, we see that the matrices $(\boldsymbol{I} - \boldsymbol{Z}_{+}\boldsymbol{Z}_{-})$ and $(\boldsymbol{I} - \boldsymbol{Z}_{-}\boldsymbol{Z}_{+})$ are invertible. In particular, we see that the inverse $(\boldsymbol{I} - \boldsymbol{Z}_{-}\,\boldsymbol{Z}_{+})^{-1}$ can be expanded as a power series $\sum_{k=0}^{\infty}(\boldsymbol{Z}_{-}\,\boldsymbol{Z}_{+})^k$ and

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{Z}_{-} \\ \boldsymbol{Z}_{+} & \boldsymbol{I} \end{pmatrix}^{-1} = \begin{pmatrix} (\boldsymbol{I} - \boldsymbol{Z}_{-}\boldsymbol{Z}_{+})^{-1} & -\boldsymbol{Z}_{-}(\boldsymbol{I} - \boldsymbol{Z}_{+}\boldsymbol{Z}_{-})^{-1} \\ -\boldsymbol{Z}_{+}(\boldsymbol{I} - \boldsymbol{Z}_{-}\boldsymbol{Z}_{+})^{-1} & (\boldsymbol{I} - \boldsymbol{Z}_{+}\boldsymbol{Z}_{-})^{-1} \end{pmatrix}.$$

Substituting the boundary conditions (5) in (11) and (12), we find that

$$\boldsymbol{\Psi}^{+}(u) = \big(\exp(\boldsymbol{Q}_{+}(K-u)) - \boldsymbol{\eta}_{-}\exp(\boldsymbol{Q}_{-}u)\boldsymbol{Z}_{+}\big)\sum_{k=0}^{\infty}(\boldsymbol{Z}_{-}\boldsymbol{Z}_{+})^{k}. \tag{13}$$

Analogously, we find

$$\boldsymbol{\Psi}^{-}(u) = \big(\boldsymbol{\eta}_{-}\exp(\boldsymbol{Q}_{-}u) - \exp(\boldsymbol{Q}_{+}(K-u))\boldsymbol{Z}_{-}\big)\sum_{k=0}^{\infty}(\boldsymbol{Z}_{+}\boldsymbol{Z}_{-})^{k} \tag{14}$$

Now we translate back the results for the fluid model (J, U') to our original model (1) for U. Note that the initial phase distribution of the interarrival times is given by the vector $\boldsymbol{\alpha}$. Thus, we have the following result expressing the probabilities $\Psi_{\delta}^{\pm}$ in terms of the Wiener-Hopf factorization $(\boldsymbol{\eta}_{\pm}, \boldsymbol{Q}_{\pm})$ of the matrix $\boldsymbol{K}_{\delta}$ given in (6).

Theorem 1. *Under the model (1) the probabilities $\Psi_{\delta}^{\pm}$ are given by*

$$\Psi_{\delta}^{+}(u,K) = \boldsymbol{\alpha}\boldsymbol{\Psi}^{+}(u)\mathbf{1}_{m} \qquad \Psi_{\delta}^{-}(u,K,y) = \boldsymbol{\alpha}\boldsymbol{\Psi}^{-}(u)\exp(\boldsymbol{B}y)\mathbf{1}_{n}$$

where $\boldsymbol{\Psi}^{\pm}$ are given in (13) and (14) with $\boldsymbol{Z}_{\pm} = \boldsymbol{\eta}_{\pm}\exp(\boldsymbol{Q}_{\pm}K)$.

Remarks. (i) For $K \to \infty$ we see that $\Psi_{\delta}^{+}(u,K)$ tends to zero (as $\boldsymbol{Q}_{\pm}$ are negative definite) and that $\Psi_{\delta}^{-}(u,K,y)$ tends to $\Psi_{\delta}^{-}(u,\infty,y)$ given by

$$\mathbb{E}_{u}[\mathrm{e}^{-\delta\tau}; U_{\tau} \le -y] = \boldsymbol{\alpha}\boldsymbol{\eta}_{-}\exp\big((\boldsymbol{B}+\boldsymbol{b}\boldsymbol{\alpha}\boldsymbol{\eta}_{-})u + \boldsymbol{B}y\big)\,\mathbf{1},$$

where $\tau = \inf\{t \ge 0 : U_t < 0\}$ is the time of ruin. This expression is well known in the literature (e.g. [3], [9]).

(ii) In our model throughout, we assumed the discounting δ to be positive. Results for $\delta = 0$ can be obtained by approximation, that is, $\Psi_{0}^{+}(u,K)$ is equal to the limit as δ tends to zero of $\Psi_{\delta}^{+}(u,K)$. The same holds for $\Psi_{0}^{-}(u,K,y)$.

(iii) The formulae can be probabilistically interpreted as follows. Multiplying out the summations in (13) and (14) one can write Ψ_{δ}^{+} and Ψ_{δ}^{-} as a series terms of which the sign alternates. Using the probabilistic interpretation of the Wiener-Hopf factors, one can give an inclusion-exclusion argument for the formulae. For example, for $\Psi_{\delta}^{+}(u,K)$, conditioned $U_0 = u$, the first three terms are respectively the probability that U crosses the level K, that U downcrosses 0 and then crosses K and that U crosses K, then downcrosses 0 and finally upcrosses K again.

(iv) The approach followed here can be used to solve associated exit problems, like for example the passage problem for U reflected at its infimum. Furthermore, upward jumps or Erlangian killing (to approximate the finite time ruin probabilities as in [7]) can be incoporated. See the forthcoming paper [10].

References

[1] Sparre Andersen, E. (1957), On the collective theory of risk in the case of contagion between claims, *Transactions of the XVth International Congress of Actuaries* **II**, 219-229.

[2] Asmussen, S. & Rolski, T. (1991), Computational Methods in risk theory: a matrix-algorithmic approach, *Insurance: Mathematics and Economics* **10**, 259-274.

[3] Asmussen, S. (1992), Phase-type representations in random walk and queueing problems, *Ann. Probab.* **20**, 772-789.

[4] Asmussen, S. (2000), Matrix–analytic models and their analysis, *Scand. J. Statist.* **27**, 193–226.

[5] Asmussen, S. (2000), *Ruin Probabilities* (World Scientific).

[6] Asmussen, S. & Kella, O. (2000), A multi–dimensional martingale for Markov additive processes and its applications, *Adv. Appl. Probab.* **32**, 376–393.

[7] Asmussen, S., Avram, F. & Usabel, M. (2001), Erlangian approximations for finite–horizon ruin probabilities. To appear in *Astin Bulletin*

[8] Avram, F. & Usabel, M. (2001), Finite time ruin probabilities with one Laplace inversion, Submitted to *Insurance: Mathematics and Economics*

[9] Avram, F. & Usabel, M. (2001), Ruin probabilities and deficit for the renewal risk model with phase–type interarrival times, Submitted to *Scand. Act. J.*

[10] Avram, F. (2002), An ordinary differential equations approach for finite time ruin probabilities, including interest rates, Working paper.

[11] Barlow, M.T., Rogers, L.C.G. & Williams, D. (1980), Wiener-Hopf factorization for matrices, in: *Seminaire de Probabilites* **XIV**, Lecture Notes in Math. **784** (Springer, Berlin) pp. 324–331.

[12] Cramèr, H. (1955), *Collective Risk Theory.* Jub. Volume of F. Skandia.

[13] London, R.R., McKean, H.P., Rogers, L.C.G. & Williams, D. (1982), A martingale approach to some Wiener-Hopf problems II, in: *Seminaire de Probabilites* **XVI**, Lecture Notes in Math. **920** (Springer, Berlin) pp. 68–90.

[14] Mitra, D. (1988), Stochastic Fluid Models, in: P. J. Courtois and G. Latouche (eds.) *Performance* **87** (Elsevier, North-Holland) pp. 39-51.

[15] Neuts, M.F. (1981), *Matrix-Geometric Solutions in Stochastic Models.* (Johns Hopkins University Press, Baltimore. London).

[16] Rogers, L.C.G. (1994), Fluid models in Queueing theory, *Ann. Appl. Probab.* **4**, 390-413.