On Large Deviations
for Polyhedral Regulated Brownian Motion

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Abstract We show below that the two dimensional RBM variational problem has a local structure which reduces its solution to a finite choice problem. This is illustrated by obtaining a complete description of possible tail behaviors for the case of the two dimensional quadrant.

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1 Introduction

The "regulated Brownian motion" approximation for queuing networks promoted by J.M. Harrison has become one of the most useful tools for obtaining insights in the behavior and optimization of networks. Some of the pioneering papers in this field are [28],[16],[21],[29], [20], [32], [33], [18], [19]. Unfortunately the stationary distribution of regulated Brownian motion may be not be obtained explicitly, except for a few particular symmetry cases (see for example [16]) or cases when it has product form (see [19] or Example 1 below).

There are however important applications in which just the asymptotic behavior of the stationary distribution (the "tail" behavior) is enough to yield valuable qualitative insights (see for example [15] for the setting of the optimal base stock levels in inventory problems). The tail behaviour is in principle computable by the large deviations variational principle (see for example [14], [31] [10]) which reduces the problem to the computation of an optimal "escape path".

The presence of boundaries however gives rise to both foundational and computational challenges. While the foundational challenges have not yet been resolved (see though [34] and [3] for state of the art reviews on criteria for positive recurrence and [26] for conditions which ensure the validity of the large deviations principle for RBM's), the computational ones are no less interesting.

The general problem is to determine the tail behaviour (i.e. the asymptotics of the stationary distribution) for stationary transition Markov processes with polyhedral state space, transition mechanisms which are possibly different in the interior and on each part of the boundary and whose stationary distribution viewed from far away concentrates around the origin.

It became clear from the work of several authors including Knessl and Tier, Ignatyuk, Malyshev and Scherbakov and Majewski that the tail behavior of such processes has in fact a rather simple form.

Each boundary may have a (possibly empty) adjoining "influence domain", i.e. a region in which the tail behaviour is determined solely by that boundary's parameters. There may be a "no influence" domain (in which all the boundary parameters become irrelevant asymptotically) and there may be also domains which "feel the influence" (i.e. boundary parameters) of any given subset of boundary regions.

The above follows from the large deviations variational principle which states that at points X far away from the origin the stationary probability is more and more determined by a unique "escape" path (the most likely travel path from the origin to X). Furthemore an escape path must be piecewise linear, since it must have constant velocity both in the interior and on each part of the boundary.

This leads to a polyhedral partition of the interior of the state space in regions characterized by the different structure of their escape path. For example, direct escape paths produce the "no influence" domain and escape paths which use only a certain part of the boundary give rise to the influence domain of that part.

In this paper we illustrate the above program in the case of two dimensional RBM's. Taking advantage of the simple quadratic large deviations functional we elucidate the local structure of the escape paths. We find that each boundary segment has an associated "entrance velocity" Z and "exit velocity" Z^* (which are reflections of each other with respect to the scalar product induced by the covariance matrix) such that:

1. An optimal escape path may hit from the interior and then follow a certain boundary segment only at its associated "entrance velocity" Z.

2. An optimal escape path may follow a certain boundary segment and then leave towards the interior only at the associated "exit velocity" Z^*

3. A boundary segment may be a profitable intermediate "speedway" for reaching interior points iff its entrance velocity Z is directed towards the outside of the state space (and Z^* towards the interior), as opposed of being directed along the boundary.

The above results provide a complete description of the local behavior of escape paths. It seems that the global problem of choosing among the various possible subsets of "boundary speedways" may only be resolved on a case by case basis, as illustrated for the two dimensional quadrant in Section 4. RBM's and the corresponding variational problem are presented in Section 2. Our results are given in Section 3, with the proofs deferred to Section 5.

We note that our complete characterization of the tail behavior of the two dimensional regulated Brownian motion in the quadrant unifies the exact results of [16] and [19] as well as the perturbation results of Knessl and Tier [23], [24], [25] (which used however a different, nonvariational approach).

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2 Regulated Brownian Motion and Large Deviations

We review now briefly some known facts about RBM's. For more details see [34] and [12].

Let S denote a "simple" polyhedral state space $\{X'N_i \geq c_i, \quad \forall i \leq K\}$. (N_i here represent the inner normals and "simple" means that each boundary of dimension k is the intersection of exactly $n-k$ hyperplanes.)

Definition: Semimartingale n dimensional M, A, R regulated Brownian motion on S is the process defined by the equation

$$X(t) = W(t) + \sum^{K} l_i(t) R_i \tag{1}$$

where $W(t)$ is n dimensional Brownian motion with constant drift M and covariance matrix A, $(l_1(t), \ldots, l_K(t))$ denote the local times of the process on the boundaries, and R_i represent directions of "pushes from the boundary $X'N_i = c_i$" (which are destined to keep the process inside S). Informally $X(t)$ above maybe viewed as relative coordinates with respect to a point in the moving state space S, with S being pushed in the direction opposite to R_i whenever $W(t)$ reaches the boundary (a fly in a moving cage).

One seminal result of Harrison and Reiman was the simultaneous construction of $X(t)$ and $(l_1(t), \ldots, l_K(t))$ provided in [17] by a pathwise function of $W(t)$ (called nowadays Skorohod mapping). The behaviour of this mapping is best understood when applied to linear paths $W(t) = Z\,t + W_0$.

In the simplest case, upon hitting a boundary B_I defined by $\{X'N_i \geq c_i, i \in I\}$ (note that for this we must have $Z'N_i < 0, \quad \forall i \in I$) the linear path will proceed through B_I with a **regulated velocity** $\tau_B(Z)$:

$$\tau_B(Z) = Z + \sum_{i \in I} x_i R_i \tag{2}$$

where x_i, the "push rates" from boundary i, are determined by the system $\{N_i'\ \tau_B(Z) = 0, \quad \forall i \in I\}$ and must be strictly positive.

It may also happen that after hitting B_I the linear path will proceed to a different region of the boundary of smaller dimensionality (for example when going from an edge to one of the faces defining that edge). In this case we will still have $N_i' \tau_B(Z) = 0$ and x_i strictly positive for the directions in which the path stays on the boundary, while in the others we will have $N_i' \tau_B(Z) > 0$ and $x_i = 0$.

In conclusion, (2) has to be interpreted as a local linear complementarity problem at each boundary B_I (the complementary variables being the "pushes" x_i and the normal components of the regulated velocity $\tau_B(Z)$). This complementarity problem has solutions iff:

Assumption S: For each boundary B_I there is a linear positive combination V of the corresponding regulation directions pointing to the interior of the state space (i.e. such that $V'N_i > 0 \quad \forall i \in I$). Furthemore this condition ensures that the RBM defined by (1) may indeed be constructed (see [12], Theorem 1.3 and also [30], [1] for earlier references).

Remark: For hyperplane boundaries, Assumption (S) simply means that $R_i'N_i > 0$ and w.l.o.g. we will assume from now on that $R_i'N_i = 1$.

In conclusion, we will use a velocity projection operator τ defined by $\tau(Z, X) = Z$ for X in the interior of the state space and by $\tau(Z, X) = \tau_{B_I}(Z)$ when $X \in B_I$, $\{Z'N_i < 0, \quad \forall i \in I\}$ and all the "pushes" defined by (2) are strictly positive (for Z not satisfying these conditions X may not spend positive time on the boundary B_I and the projection need not be defined at all). From now on we simplify the notation by letting $\tau(Z, X) = \tilde{Z}$.

2.1 Large deviations for regulated Brownian motion

Essentially, the asymptotic behaviour of the stationary density at a given point X is found by solving the variational problem defined below of finding the most likely velocity Z for escaping from the origin to X with the extra provision that while on the boundary the particle moves with a regulated velocity $\tilde{Z}$.

Variational Problem:

$$\begin{aligned} q(X) &= \min_{X(t), 0 \le t \le T} \int_0^T \frac{1}{2}(Z_t - M)'A^{-1}(Z_t - M)\, dt \qquad (VP) \\ \dot{X}(t) &= \tilde{Z}_t \\ X(0) &= (0,0) \\ X(T) &= X \\ X(t) &\ge (0,0) \end{aligned}$$

Note:The integrand in (VP) is the well-known "loglikelihood" of the Brownian path and will be refered to in the future as the cost of a path.

Conjecture: For any positive recurrent RBM for which the spectral radius of the matrix $I - R$ is less than 1 the solution $q(X)$ of the variational problem above yields the tail behavior of the stationary distribution $p(X)$, i.e.

$$\lim_{n \to \infty} -\frac{log[p(nX)]}{n} = q(X) \qquad (3)$$

This conjecture has as yet been established only in special cases like that of triangular reflection matrices (see [26]) and that of the two dimensional quadrant. It is based on the fact that the Lifschitzness of the Skorohod mapping which holds under the condition above (see [11], Sections 4.2 and 5.5.2) is quite likely to imply the large deviations principle.

Our focus in this paper is the analytic solution of a two dimensional version of the variational problem above.

We will assume:

Assumption A: The state space S is an unbounded two dimensional polyhedron containing the origin on its boundary.

2.2 Positive Recurrence

It was shown by Dupuis and Williams [12] (see also [7] for a powerful generalization) that RBM is positive recurrent iff its fluid limit, which in the interior of the state space is simply a linear path with constant velocity M, ends at the origin from any initial starting point. For the case of two dimensional polyhedra this leads to a simple criterium.

Indeed, let us note that for any fixed linear boundary B with unit directing vector E and regulation vector R the boundary projection (2) may be written as

$$\tau_B(Z) = (Z'P)\,E \tag{4}$$

where P is the unique vector satisfying

$$\begin{aligned} P'\,R &= 0 \\ P'\,E &= 1. \end{aligned}$$

(Just note that this expression satisfies $\tau_B(R) = 0$ and $\tau_B(Z) = Z$ when $Z = tE$).

Let us choose now on each boundary segment a unit vector E_i pointing in the direction we would take to get away from the origin (following the boundary).

Let B_r denote the "reachable" part of the boundary, i.e. the union of all the boundary segments satisfying $M'N_i < 0$ (in the figure above B_r includes all the segments except $E5$).

The fluid limit stability criterion reduces in our case to:

Assumption B: 1) M is not a "ray" of the state space (i.e. there is no point X in S such that $X + \lambda M$ stays in S for arbitrarily large λ.)

2) The origin belongs to the reachable part of the boundary B_r.

3) For any segment B_i in the reachable part of the boundary B_r the "fluid boundary velocity" to be denoted by $\tilde{M}_i' = M'P_i$ is negative (and so the fluid limit proceeds towards the origin after hitting any segment).

The Fluid Flow of a Positive Recurrent RBM

Remark: In the case of the two dimensional quadrant Assumption (B) reduces to the negativity of

$$\begin{aligned} \tilde{M}_1' = &\quad m_2 + m_1 r_1 \\ \tilde{M}_2' = &\quad m_1 + m_2 r_2 \end{aligned}$$

a criterion first obtained by Hobson and Rogers [21].

We will actually assume below for the purpose of simplifying the statements that:

Assumption B3' The "fluid boundary velocity" $\tilde{M}_i'$ is negative on all the boundary segments.

It is possible to show that for the first boundary segment adjoining B_r Assumption (B3') is actually a consequence of Assumption (S). This convenience assumption is thus less restrictive than it looks.

Finally we note that the criteria for positive recurrence via fluid limits has not yet yielded necessary and sufficient algebraic conditions in more than two dimensions.

We turn now to the solution of the two dimensional variational problem.

3 The Speedway Problem

Let us denote by $< V, W >$ the scalar product $VA^{-1}W$ and by $\|V\|$ the corresponding norm.

We show below that whenever an escape path follows a certain boundary segment B it does so at an unique optimal velocity velocity Z_B (which is the same as the entrance velocity of that boundary). It also turns out that the norm $\|Z_B\| = \|M\|$ is actually to be independent of B. Our variational problem reduces thus to a problem of making up an optimal routing from 0 to X from boundary and interior segments taking advantage of the fact that for certain boundary segments B the regulated velocity $\tilde{Z}_B$ is greater than $\|M\|$. It is convenient to think of it as of the familiar "speedway choice problem".

Propositions 1 and 2 below determine the optimal velocity Z for single segment escape paths belonging to the interior and boundary respectively.

Propositions 3 and 4 determine the optimal breaking angle for two segment paths. Beyond two segments, the problem is essentially combinatorial (of choosing among a finite set of possibilities) and we can only solve it in special instances.

3.1 Interior Escape

When the escape path is formed from one interior segment Z must be proportional to X and the variational problem (VP) reduces to the scalar minimization

$$q_0(X) = min_a \frac{< a\, X - M, a\, X - M >}{2|a\, X|} \qquad \text{(DIRESC)} \qquad (5)$$

where we let $| \quad |$ denote Euclidean norm. Setting the first derivative to 0 yields:

Proposition 1 *The most likely velocity of escape along the direct path to X is $Z = \|M\| \frac{X}{\|X\|}$ and the corresponding minimal cost is*

$$q_0(X) = \|M\| \, \|X\| - < M, X > \qquad (6)$$

Note: We will also need later the minimal cost along a direct path from tE to X; by homogeneity, this is

$$q_0(X - tE) = \|M\| \, \|X - tE\| + < M, E > t - < M, X > \tag{7}$$

3.2 Single Segment Boundary Escape

We consider now escape paths formed from one segment along a boundary with unit directing vector E, inner normal N and regulation direction R.

Since the projected velocity along this boundary is $\tilde{Z} = (Z'P)E$ (where P denotes the unique vector perpendicular to R and satisfying $P'E = 1$) the variational problem along this boundary is:

$$\begin{aligned} min_Z \quad & \frac{(Z-M)'A^{-1}(Z-M)}{2Z'P} \qquad \text{(BOUNDESC)} && (8) \\ Z'P \;&>\; 0 && (9) \\ Z'N \;&\leq\; 0 && (10) \end{aligned}$$

The first constraint above reflects the fact that we always need to travel away from the origin (the opposite direction may be achieved anyway at 0 cost by using $Z = M$).

Proposition 2 *a)The solution Z of the* **unconstrained** *minimization problem (BOUNDESC) is*

$$Z = M + kAP \tag{11}$$

where

$$k = \frac{-2M'P}{P'AP} = \frac{-2M'P}{b} \tag{12}$$

and $b = P'AP$ is a positive constant.

b)Z defined above satisfies

$$\begin{aligned} \|Z\| &= \|M\| && (13) \\ Z'P &= -M'P && (14) \end{aligned}$$

and thus satisfies automatically the first constraint of (BOUNDESC).

c) Along boundaries for which (11) satisfies also the second constraint strictly, the minimum cost to a point $X = tE$ is:

$$q_b(X) = k\, t =< Z - M, X > \tag{15}$$

d) For boundaries on which the optimum escape velocity satisfies $Z'N = 0$ the formula (15) above still holds but with different k and Z.

$$k = \|M\|\|E\| - < M, E > \tag{16}$$

$$Z = \frac{\|M\|}{\|E\|} E \tag{17}$$

Remarks: 1) To distinguish between cases (c) and (d) above we will use a normalized form of the second constraint:

$$f = -bZ'N \tag{18}$$

to be called from now on "reflectivity".

2) The interpretation of the positive reflectivity condition distinguishing cases (c) and (d) is that in the original stochastic problem the most likely escape happens by "bouncing" against the boundary (and using lots of regulation) in the first case and by going straight towards the destination (parallel to the boundary) in the second. We will show below that only the first type of boundaries may be profitable "speedways" (i.e. they may be good to use also for reaching interior points).

3) The formula $q_b(X) =< Z - M, X >$ holds in fact for optimal escape paths through boundaries B_I of arbitrary dimensionality (with Z denoting the escape velocity). However the constant k appearing in the unconstrained solution of (BOUNDESC) has to be replaced by a vector K representing the coordinates of the projection of Z on B_I. K satisfies the rather complicated system $K - 2\frac{|K|^2}{K'QK}QK = P'M$ where P is the projection matrix and $Q = P'AP$, which seems to make the multidimensional case more difficult.

3.3 Optimal two segment escapes

We will identify now the "optimal broken path" (OBP) among all paths starting (w.l.o.g) from the origin and formed from two segments: one along a linear boundary with directing vector E up to a point tE and a second segment through the interior, from tE to X. Note that the broken path obtained when $t = 0$ is a direct path.

We denote by R, N, f the regulation, normal and reflectivity of the boundary.

Proposition 3 *a) For linear boundaries with negative reflectivity the cost of the direct path as given in (6) is lower than that of any broken path to X.*

b) OBP's through a linear boundary with positive reflectivity may leave it in an interior point only at the unique exit velocity Z^* which is the reflection of the entrance velocity Z with respect to the scalar product $<,>$; i.e. Z^* is the unique vector different from Z which satisfies

$$
\begin{aligned}
< Z^*, E > &= < Z, E > && (19) \\
\|Z^*\| &= \|Z\| = \|M\| && (20)
\end{aligned}
$$

c) The total cost along an OBP is linear, given by $q(X) = QX$ where

$$
Q = A^{-1}(Z^* - M) = k\ P + 2\frac{f}{\|AN\|^2}N \tag{21}
$$

Remarks: 1) The domain of influence of a boundary with positive reflectivity may not extend beyond the path in the direction Z^* which starts at the origin (the "OBP" boundary).

2)Z^* may be obtained from the decomposition of Z with respect to the orthogonal basis E, AN

$$
Z =< Z, E > \frac{E}{\|E\|^2} + < Z, AN > \frac{AN}{\|AN\|^2}
$$

by changing the sign of the component transversal to E.

$$Z^* =< Z, E > \frac{E}{\|E\|^2} - < Z, AN > \frac{AN}{\|AN\|^2}$$

3) To determine the slope of Z^* we will also need later its Euclidean decomposition $Z^* = (E'Z^*)E + (N'Z^*)N$. We find that:

$$N'Z^* = \frac{f}{b} \tag{22}$$

$$E'Z^* = \frac{(< M, E > +k)b(a_1a_2 - a^2) + f\ E'AN}{N'AN\ b} \tag{23}$$

We consider now optimal broken paths which start with an interior segment and then proceed along some positive reflectivity boundary.

Proposition 4 *Optimal two segment broken paths can only enter a positive reflectivity boundary segment in an interior point at the velocity Z given in Proposition 1 a).*

While the local behaviour of the escape paths is completely determined by the results above, to solve speedway problems one still has to compare various types of "broken paths" moving through various parts of the boundary. This can only be achieved for individual instances of the problem. One example is provided in the next section.

4 Possible tail behaviors in the two dimensional quadrant

Let $E_i, N_i, R_i, P_i, b_i = P_i' A P_i, Z_i$ denote the corresponding characteristics of the linear boundaries $x_i = 0, i = 1, 2$ and let $\tilde{M}_i = M' P_i$ denote the "fluid boundary velocities".

Formula (18) yields here the reflectivities:

$$f_i = -b_i Z_i' N_i = -b_i M' N_i + 2(M' P_i)(P_i' A N_i) = -m_i b_i + 2\tilde{M}_i(a_i r_i + a) \tag{24}$$

Corollary 1 If neither boundary has positive reflectivity *then $q(X)$ is given by the solution of (DIRESC)*

$$q_0(X) = \|M\|\|X\| - < M, X > \tag{25}$$

throughout the quadrant.

Let now Z_i^* denote the leaving velocity of the OBP's from the boundary $x_i = 0$ and let s_i be their slope. From Remark 3 following Proposition 3 we find that the slopes are

$$s_1 = \frac{y_1}{a_1 f_1}$$
$$s_2 = \frac{a_2 f_2}{y_2}$$

where

$$\begin{aligned} y_1 &= (< M, E_1 > + k_1) b_1 (a_1 a_2 - a^2) + f_1 \ E_1' A N_1 \\ &= -2a \ a_2 \ m_1 + 4a^2 \ m_2 - a_1 \ a_2 \ m_2 - 2a_1 \ a_2 \ m_1 \ r_1 + 4a \ a_1 \ m_2 \ r_1 + a_1^2 \ m_2 \ r_1^2. \end{aligned}$$

and y_2 is defined analogously.

Corollary 2 If precisely one of the boundaries, say $x_1 = 0$, has positive reflectivity *then the tail behaviour will be given by the linear function*

$$q_1(X) = \frac{2(f_1 - a_1 r_1 \tilde{M}_1)}{a_1 b_1} x_1 - \frac{2\tilde{M}_1}{b_1} x_2 \tag{26}$$

This formula will hold throughout the quadrant if the corresponding OBP slope s_1 is negative. If however s_1 is positive, the formula changes to the minimum direct escape cost (6) beyond the OBP boundary.

It remains now only to look at the case when both f_i are positive. In this case we have to decide first whether the two OBR cones overlap, which will happen if at least one of the slopes s_i is negative, or if both are positive and $s_1 - s_2 < 0$.

The separation boundary between the two OBR cones is decided by considering the difference of the linear costs near each boundary, which we write as $q_2(X) - q_1(X) = p_1x_1 - p_2x_2$.

We give below the unfortunately rather complicated formulas of $s_1 - s_2$ and of p_i. They contain both the factor $j = 2a + a_1r_1 + a_2r_2$ which implies that when $j = 0$ the same exponential density will satisfy both boundary conditions.

$$p_1 = 2j\,\frac{f_2 - \tilde{M}_2\,j}{a_1b_1b_2} \tag{27}$$

$$s_1 - s_2 = 2\,j\,\alpha(M)\,\frac{a_1a_2(1 - r_1r_2) - a\,j}{a_1f_1y_2} \tag{28}$$

Here $b_1 = a_2 + 2ar_1 + a_1r_1^2$, $b_2 = a_1 + 2ar_2 + a_2r_2^2$ and $\alpha(M) = a_2m_1^2 - 2am_1m_2 + a_1m_2^2$ are positive constants of proportionality.

Corollary 3 If both boundaries have positive reflectivity *then:*

a) If the OBP cones overlap and if exactly one of the two coefficients p_i is positive then the respective family of OBP's has lower cost than the other throughout the quadrant.

b) If the OBP cones overlap and both the coefficients p_1 and p_2 are negative than a partition of the quadrant by the line $p_1x_1 - p_2x_2 = 0$ occurs.

c) If the OBP cones do not overlap ($s_i > 0, i = 1, 2$ and $s_1 - s_2 > 0$) then $q(x)$ is given by the two linear forms q_i near the respective boundaries and by (6) in the sector in between.

4.1 Particular cases

Example 1: Continuous Jackson network When $j = 0$ we have $p_1 = p_2 = 0$ and $s_1 = s_2$, which imply $q_2(X) = q_1(X) = q(X)$. In this case by coincidence, the two exponential expressions satisfying the two boundary conditions coincide. This is the "skew symmetric" case of [19], a continuous analog of the famous discrete Jackson networks.

Example 2: Normal reflection

We carry now the computation further for the case of normal reflection. While the

Correlation Range	
$a < -S/\sqrt{2}$	$\{12\} \longleftarrow F_1\{2\}Y_2 \longrightarrow \{23\}Y_1 \longrightarrow \{23\} \longleftarrow F_2\{3\}$
$-S/\sqrt{2} < a < -S/2$	$\{12\}Y_2 \longrightarrow \{12\} \longleftarrow F_1\{23\} \longleftarrow F_2\{3\}Y_1 \longrightarrow \{3\}$
$-S/2 < a < 0$	$\{12\} \longleftarrow Y_1\{12\}Y_2 \longrightarrow \{12\} \longleftarrow F_2\{1\} \longleftarrow F_1\{3\}$
$0 < a < S/2$	$\{23\}F_1 \longrightarrow \{123\} \longleftarrow F_2\{13\} \longleftarrow Y_2\{13\} \longleftarrow Y_1\ \{1\}$
$S/2 < a < S/\sqrt{2}$	$\{23\}Y_1 \longrightarrow \{23\} \longleftarrow F_2\{3\}F_1 \longrightarrow \{13\} \longleftarrow Y_2\{3\}$
$S/\sqrt{2} < a$	$\{23\} \longleftarrow F_2\{3\}Y_1 \longrightarrow \{3\} \longleftarrow Y_2\{3\}F_1 \longrightarrow \{13\}$

Table 1: Normal Reflection

marginals are trivial in this case (they are unaffected by the existence of the other component), there is no major simplification in the joint distribution.

It is convenient to divide the analysis by the sign of the three quantities: $a, \Delta_1 = a_1a_2 - 2a^2, \Delta_2 = a_1a_2 - 4a^2$. Below is a summary of various equalities and proportionalities useful in this case

$$\begin{aligned}
j &= 2a \\
s_1 - s_2 &\approx \frac{a\Delta_1}{f_1y_2} \\
y_1 &= p_2 - \Delta_2 m_2 \\
p_1 &\approx -am_2 \\
f_1 &= -a_2m_1 + 2am_2
\end{aligned}$$

Symmetric relations hold for the other boundary.

Finally, we will need to specify the position of the direction of the drift m_2/m_1 with respect to the four numbers $F_1 = \frac{a_2}{-2a}, F_2 = \frac{-2a}{a_1}Y_1 = \frac{2aa_2}{\Delta_2}, Y_2 = \frac{\Delta_2}{2aa_1}$ where f_1, f_2, y_1, y_2 change sign respectively.

Let $S = \sqrt{a_1a_2}$. Table 1 below, divided in six cases according to the size of a with respect to S, indicates the relative positions of these numbers, as well as the sign of the corresponding factor afterwards;thus $\longleftarrow Y1$ would mean that y_1 is positive before Y_1. We indicate also the type of stationary distribution: $\{1\}$ indicates only OBP's from $x_1 = 0$, $\{1\ 2\ 3\}$ indicates all three regions, etc

5 Solution of the Variational Problem

Proof of Proposition 2

a) Recall the minimization problem (BOUNDESC) (8)

$$min_Z \frac{(Z-M)'A^{-1}(Z-M)}{Z'P} \tag{29}$$

$$Z'P > 0 \tag{30}$$

$$Z'N \leq 0 \tag{31}$$

Differentiating to find the unconstrained minima we find that Z must be of the form

$$Z = M + kAP \tag{32}$$

with $k = 0$ or

$$k = \frac{-2M'P}{P'AP} = \frac{-2M'P}{b} \tag{33}$$

where we let b denote the positive constant $P'AP$.

The first solution $Z = M$ may never be optimal since by the stability Assumption (B3) M it may not satisfy simultaneously the constraints (30) and (31).

b) We may check that Z defined in (32) satisfies the relations (13) and (14) by plugging the value $k = \frac{-2M'P}{P'AP}$.

c) Plugging (32) in the cost function (29) yields $q_b(E) = k$. Noting further that k defined in (33) satisfies $k =< Z - M, E >$ we obtain for $X = tE$

$$q_b(X) = kt =< Z - M, X > .$$

d) When the optimum satisfies $Z'N = 0$ we have a particular case of direct escape covered in Proposition 1.

We turn now to the proof of Proposition 3. Recall that we must determine the optimal broken paths which follow a boundary up to tE before turning towards their destination X.

Differentiating the total cost $c(t, X) = q_b(tE) + q_0(X - tE)$ we are lead to studying the properties of the function $\frac{c'(t)}{\|M\|} = \alpha'(t) + l$ where $\alpha(t) = \|X - tE\|$ and l is the sum of the linear coefficients in the boundary and direct costs:

$$l = (k+ < M, E >)\|M\|, \tag{34}$$

Using the two formulas for k (12), (16) given in Proposition 2 a) and d) we find that

$$\begin{array}{llll} & l = \|E\| & \text{if} & f \leq 0 \\ l = \dfrac{< Z, E >}{\|M\|} = & \dfrac{< Z, E >}{\|Z\|} & \text{if} & f > 0. \end{array}$$

Finally let us note that the derivative $\alpha'(t)$ may be written as: $-\frac{<X-tE,E>}{\|X-tE\|}$.
Summarizing, the derivative of the total cost $c(t, X)$ is:

$$c'(t, X) = \|E\| - \frac{<X-tE,E>}{\|X-tE\|} \quad if \quad f < 0 \tag{35}$$

$$c'(t, X) = \frac{< Z, E >}{\|Z\|} - \frac{<X-tE,E>}{\|X-tE\|} \quad if \quad f > 0 \tag{36}$$

Proof of Proposition 3

We check first that the cost is a convex function of t by differentiating the formula above.

a) In the case of negative reflectivity boundaries we find by applying Schwarts's inequality in (35) that the derivative is positive and thus the total cost is a strictly increasing function of t whose minimum is achieved at $t = 0$ by the direct path.

b) In the case of positive reflectivity boundaries we determine the unique minimum of $c(t, X)$ as the root of (36).

We note now that the equality (36) determines the direction for $X - tE$. Indeed, letting $X - tE = aZ^*$ (where a is a normalization constant to be determined later) we find that a cancels and we get

$$\frac{< Z, E >}{\|Z\|} = \frac{< Z^*, E >}{\|Z^*\|} \tag{37}$$

which leaves two possible directions for Z^* : that of Z and its reflection. The first is however not feasible here, because on positive reflectivity boundaries Z points towards the outside of S and X is in the inside.

To determine also the magnitude of the exit velocity we note that the segment connecting tE to X is actually a direct escape for which we have already established that the optimal escape velocity must have norm equal to $\|M\|$. If we choose thus Z^* of the same norm with Z and M it will yield not only the direction, but also the velocity of the departing path. This also has the advantage of cancelling the denominators above yielding the convenient formula (19).

c)We plug now $X - tE = aZ^*$ in the total cost formula $q_b(tE) + q_0(X - tE)$ taking into account the expressions for the costs derived in Proposition 2 c) and the Remark following Proposition 1 respectively.

$$
\begin{aligned}
q(X) &= q_b(tE) + q_0(aZ^*) =< Z - M, tE > +a(\|M\|\|Z^*\| - < M, Z^* >) \\
&= < Z - M, X - aZ^* > +a(\|M\|\|Z^*\| - < M, Z^* >) \\
&= < Z - M, X > +a(\|Z\|\|Z^*\| - < Z, Z^* >)
\end{aligned}
$$

By projecting X on AN we find that $a = \frac{b<X,AN>}{f}$ and the coefficient $(\|Z\|\|Z^*\| - < Z, Z^* >)$ simplifies to $2\frac{f^2}{b^2\|AN\|^2}$, yielding thus

$$q(X) =< kAP + 2\frac{f}{b\|AN\|^2}AN, X >= Q'X$$

where $Q = kP + 2\frac{f}{b\|AN\|^2}N$.

Finally, we may check that $M + AQ = M + kAP + 2\frac{f}{b\|AN\|^2}AN = Z + 2\frac{f}{b\|AN\|^2}AN = Z^*$.

It is reassuring at the end of this long computation to note that the linear exponent Q may also be obtained in a much easier way by looking for exponential form solutions satisfying exactly the Fokker-Planck P.D.E together with the appropriate boundary condition on the neighboring axis as given for example in equation (8.5) in in [18]).

Proof of Proposition 4: The computation is identical to that in Proposition 3, the only difference being that this time after obtaining equation (37) we have to choose its second root $Z^* = Z$.

6 Conclusions

We have showed that the two dimensional RBM variational problem has a local structure which reduces its global solution to a finite choice problem.

Obtaining further correction terms beyond the large deviations approximation is clearly feasible, as demonstrated in [25].

An especially challenging outstanding problem is the extension of the above results in more than two dimensions.

Finally, let us mention that for certain particular two dimensional problems exact (usually integral) representations for the stationary distribution have been provided (see [13], [27] and [25]) which raises the question of the relationship between the two approaches.

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